

MAXIMAL QUOTIENT RINGS
OF
PRIME NONSINGULAR GROUP ALGEBRAS

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ABSTRACT

Recent work by Goodearl and Handelman has shown that a prime, regular, right self-injective ring Q must be precisely one of the following:

- (a) a full linear ring (being either simple Artinian, or infinite dimensional full linear);
- (b) a non-simple ring with zero socle;
- (c) a directly finite, non-Artinian ring (necessarily simple); that is, an $SP(\infty)$ ring;
- (d) a simple, directly infinite ring; that is, an $SP(1)$ ring which is not a division ring.

Suppose now that Q is the maximal right quotient ring of a (necessarily prime nonsingular) group algebra KG . In this thesis we try to determine how this extra hypothesis affects the above classification.

We prove in Chapter 1 that if all the conjugacy classes of G are countable then Q is either a full linear ring or a simple, directly infinite ring.

In Chapter 2 we assume that G is locally finite. Using a dimension function on the finitely generated right ideals of KG , we show that if G is a nontrivial, locally finite group with only countable conjugacy classes then Q is simple and directly infinite. This is also true if G is the restricted symmetric group on any infinite set.

In Chapter 3 we show that if Q is a full linear ring then G contains no nontrivial, locally finite, normal

subgroups. If, in addition, G is soluble or residually finite or K has zero characteristic and G linear, then Q must be simple Artinian.

In Chapter 4 we generalize the above-mentioned result from Chapter 1 and deduce that if G is soluble then Q must be a simple ring.

Finally in Chapter 5 we study the ideals of Q and their interaction with the normal subgroups of G . We show that if G is residually finite then Q is a simple ring.

INTRODUCTION

The study of maximal right quotient rings (MRQ rings) of group algebras was begun by Burgess [5] in 1969, although he worked with general group rings. His aim was to determine when the MRQ ring of a group algebra KG is semisimple Artinian but he ran up against two main obstacles. Firstly, very little was known about general MRQ rings and their abstract nature made them hard to handle. Secondly, the standard characterizations of rings whose MRQ rings are semisimple Artinian have proved difficult to translate into statements about the structure of the field K and the group G .

One of the major advances in recent years has been Goodearl's description and classification of prime, regular, right self-injective rings. It is well known that the MRQ ring Q of a prime, right nonsingular ring R is a prime, regular, right self-injective ring. Goodearl [15] showed in 1973 that the (two-sided) ideals of Q are well-ordered by inclusion, and characterized these ideals in terms of certain cardinal numbers (in an analogous manner to Jacobson's description in [31] of the ideals of a full linear ring). Furthermore, by using this and Handelman and Lawrence's work in [23] on strongly prime rings, Goodearl and Handelman [18] showed in 1975 that Q must be precisely one of the following:

- (a) a full linear ring (being either simple Artinian
 - that is, $SP(n)$ for some integer $n > 1$ or a division ring - or an infinite dimensional full linear ring);

- (b) a non-simple ring with zero socle;
- (c) a directly finite, non-Artinian ring (necessarily simple); that is, an $SP(\infty)$ ring;
- (d) a simple, directly infinite ring; that is, an $SP(1)$ ring which is not a division ring.

In this thesis I try to apply these results to the MRQ rings of prime nonsingular group algebras and, in particular, try to determine which of the four types of ring above can occur as such an MRQ ring. Although a complete answer to this question is not found (some open questions are listed at the end of this thesis), several large classes of group algebras are found for which the above classification can be simplified. For instance, I show that if KG is a prime nonsingular group algebra, where G is either a soluble or a residually finite group, then the MRQ ring of KG is a simple ring. Thus, for such group algebras, only the possibilities (c) or (d), or the simple Artinian case of (a), can occur as MRQ rings.

An outline of the development of this thesis follows and a more detailed summary of its results is provided in the tables at the end of the thesis.

Chapter 1 takes a closer look at the Goodearl-Handelman classification (above) in the case where the ring R is countable:

Theorem 1.5. Suppose R is a countable, prime, right nonsingular ring. Then the MRQ ring of R is either a full linear ring (if R has uniform right ideals) or is simple and directly infinite (if R has no uniform right ideals). \square

Thus, in this case, only (a) and (d) of the above classification are possible. This result also holds if, instead of being countable, R is either a countable dimensional algebra over a field (Theorem 1.8) or a group algebra KG where all the conjugacy classes of the group G are countable (Theorem 1.13). This condition on the group G is satisfied, for instance, by any countable or abelian group.

In Chapter 2 group algebras of locally finite groups are considered. The results here depend on a dimension function defined on the lattice of finitely generated right ideals of a locally finite group algebra. Using this function I show that, when G is a nontrivial locally finite group, the MRQ ring of KG cannot be a full linear ring (Proposition 2.8). Theorem 1.5 then implies:

Theorem 2.9. Suppose G is a nontrivial locally finite group whose conjugacy classes are all countable. If KG is prime nonsingular then its MRQ ring is simple and directly infinite. \square

Thus, in this case, only (d) of the Goodearl-Handelman classification is possible. Theorem 2.9 is also true if, instead, G is the restricted symmetric group on any infinite

set (Theorem 2.12). Even for general locally finite groups the dimension function ensures that many properties associated with countability remain valid. The function is used, for instance, to show if KG is prime nonsingular and G is locally finite then the MRQ ring of KG has at most one proper ideal (Proposition 2.13).

Formanek [13] showed in 1974 that if N is a normal subgroup of an arbitrary group G then there is a natural embedding of the MRQ ring of the group algebra KN into the MRQ ring of the group algebra KG (where K is any field). In principle this result makes it possible to translate properties of the MRQ ring Q of KG into properties of the normal subgroups of G . This approach is adopted in chapter 3 and succeeds in proving:

Theorem 3.5. If the MRQ ring Q of the group algebra KG is a full linear ring then G contains no nontrivial locally finite normal subgroups. □

Since Handelman and Lawrence [23] have conjectured that this latter condition on G is equivalent to the group algebra KG being strongly prime, this last result suggests that Q can never be an infinite dimensional full linear ring. Although unable to verify Handelman and Lawrence's conjecture, I do show that for many classes of group G it is indeed true that Q is never infinite dimensional. This is so, for example, if G is soluble (Theorem 3.19) or residually finite (Corollary 3.11), or if K has zero characteristic and G is linear (Proposition 3.21).

The following situation arises while these results are being proved:

(*) ... N is a normal subgroup of G such that G/N is locally finite and the MRQ ring of KN is semisimple Artinian.

In this situation I study the consequences of assuming that the MRQ ring Q of KG is full linear by looking at the subring S of Q generated by KG and the MRQ ring of KN (another application of Formanek's result). When (*) holds there is a dimension function, similar to that in chapter 2, on the lattice of finitely generated right ideals of S . This function is used to show that Q cannot be infinite dimensional (Theorem 3.14). However the dimension function is also ideally suited to studying a conjecture of Brown, Lawrence and Loudon as to when S coincides with Q . Using this function I prove the following result (proved by Lawrence and Loudon [36] in the special case where KG is countable):

Proposition 3.15. Suppose N is a normal subgroup of G such that G/N is locally finite and suppose KG is nonsingular. The MRQ ring Q of KG is generated by the subrings KG and the MRQ ring of KN if and only if either N has finite index in G or Q is semisimple Artinian. □

In chapters 4 and 5 the study of the MRQ ring of a general prime nonsingular group algebra is resumed. Firstly the group algebra result from chapter 1 is generalized to obtain:

Theorem 4.3. Suppose N is a normal subgroup of G whose conjugacy classes are all countable. Suppose KG is prime nonsingular and KN contains no uniform right ideals. Then the MRQ ring Q of KG is directly infinite and if I is a proper ideal of Q then $I \cap KN = 0$. \square

Using Zalessky's well known intersection theorem for soluble group algebras (see [54]), I then deduce:

Theorem 4.10. Suppose G is a soluble group such that KG is prime nonsingular. Then either KG is strongly prime or the MRQ ring of KG is simple and directly infinite. \square

In particular, it follows (as stated earlier) that the MRQ ring of KG in this last result is always a simple ring (Corollary 4.11).

Finally in chapter 5 the case of residually finite groups is considered. As a preliminary some of Goodearl's results on the ideals of prime, regular, right self-injective rings are generalized to the non-prime situation (Theorem 5.4). The methods used in chapter 3 to study the full linear case for residually finite group algebras can then be adapted to the general case. It follows, as stated earlier, that the MRQ ring of a prime, nonsingular group algebra of a residually finite group is always a simple ring (Corollary 5.12).

Some of the results in this thesis (mainly from the first three chapters) will appear shortly in the papers [24] [25] and [26].

PRELIMINARIES

We list here our basic terminology and notation, and recall the background results we shall need in this thesis. For the most part we shall use this material without any further comment.

1. RINGS.

Let R be a ring (always associative with identity) and let M be a right R -module.

We refer the reader to Faith [9] or Lambek [34] for the following notions and their basic properties: large (or essential) submodules of M (see [9], p.13 or [34], p.101), essential extensions of M (see [9], p.13), the injective envelope of M (see [9], p.19), closed submodules of M (see [9], p.15), rational extensions of M (see [9], p.58) and dense right ideals of R (see [34], p.96).

If X is a subset of M we denote the right annihilator of X in R by $\underline{r}(X)$ or $\underline{r}_R(X)$. Similarly if X' is a subset of a left R -module M' we write $\underline{\ell}(X')$ or $\underline{\ell}_R(X')$ for the left annihilator of X' in R .

The singular submodule of M is the set $Z(M) = \{m \in M : \underline{r}(m) \text{ is a large right ideal of } R\}$; see [9], pp.46-7. If $Z(M) = 0$ we say that M is nonsingular. We say that R is right nonsingular if $Z(R_R) = 0$.

Suppose S is a ring containing R as a subring. Following Utumi we say that S is a right quotient ring of R if S is a rational extension of R (as right R -modules). If R is right nonsingular this is equivalent to R being a large

right R -submodule of S (which is the definition of 'right quotient ring' used by Johnson [32]). We say that a right quotient ring S of R is a maximal right quotient ring (or an MRQ ring) of R if, for any right quotient ring T of R , there is a ring monomorphism of T into S which extends the identity map on R . We then have (see [9], pp.64-66):

Result. Every ring R has an MRQ ring. Any two MRQ rings of R are isomorphic via a map which extends the identity map on R . □

We shall denote the MRQ ring of R by $Q(R)$. The first part of the next result is proved in [9], p.69 and in [34], pp.106-7; the second part in [9], p.70.

Result. If R is right nonsingular then $Q(R)$ is (von Neumann) regular and right self-injective. Conversely if R has a regular right quotient ring then R is right nonsingular. □

Regular right self-injective rings and nonsingular injective modules are studied by Goodearl and Boyle in [17].

We say that M is uniform if it is nonzero and is an essential extension of each of its nonzero submodules. We say that M has finite right uniform dimension (or is finite dimensional) if M contains no infinite families of nonzero independent submodules. If M is finite dimensional there is a finite family of independent uniform submodules of M whose sum is large in M .

If R is right nonsingular and U is a uniform right ideal of R then the unique closed essential extension U' of U in R (see [9], p.61) is a minimal closed right ideal of R . Conversely a minimal closed right ideal of R is a uniform right ideal. From [9], p.73 we have:

Result. Suppose R is prime and right nonsingular. Then $Q(R)$ is a full linear ring if and only if R has a uniform right ideal. □

For the next result see Corollary 2 of [9], p.76.

Result. $Q(R)$ is semisimple Artinian if and only if R is right nonsingular and R has finite right uniform dimension. □

Suppose S is a ring containing R as a subring. We say that S is a classical right quotient ring of R if every non-zero-divisor of R is a unit in S and if each element of S is of the form ab^{-1} where $a, b \in R$ and b is a non-zero-divisor. A proof of the following result (due to Goldie) may be found in [9], p.80.

Result. $Q(R)$ is a semisimple Artinian classical right quotient ring of R if and only if R is right nonsingular, has finite right uniform dimension, and is semiprime. □

Throughout this thesis we shall call two-sided ideals of R simply ideals of R . We say that R is a simple ring if

the only ideals of R are 0 and R .

If $a \in R \setminus 0$ a right insulator for a in R is a subset X of R such that $\underline{r}(aX) = 0$. We say that R is (right) strongly prime if each nonzero element of R has a finite right insulator in R or, equivalently, if each nonzero ideal of R contains a finitely generated left ideal with zero right annihilator in R . Any simple ring is strongly prime. Strongly prime rings are studied in Handelman and Lawrence [23].

A (right) strongly prime ring R is said to be bounded strongly prime if there is an integer m such that each nonzero element of R has a right insulator of at most m elements. If n is the least such integer m we say that R is (right) $SP(n)$. If no such m exists we say that R is $SP(\infty)$.

We say that R is directly finite if, for any $a, b \in R$, $ab = 1$ implies $ba = 1$. Otherwise R is directly infinite. We say that M is directly finite if $\text{End}_R M$ is a directly finite ring.

If A, B are right R -modules and there is an R -monomorphism $A \rightarrow B$ we write $A \lesssim B$ and say that A is subisomorphic to B .

2. GROUPS.

In general we shall, without further comment, follow the terminology and notation of Robinson [49] for groups. Our one exception to this rule is our definition of CC-groups (see Chapter 1, p.13 below) which is considerably more general than that of Robinson (p.127 of [49]). Indeed, in our terminology, G is a CC-group if and only if, for any

$x \in G$, $\langle x \rangle^G$ is countable (cf. Theorem 4.36 of [49]).

Let H be a subgroup of G . We refer the reader to Robinson [49], Chapter 1, for the notions of a series, and of an ascending series, between H and G .

A class of groups C is a class, in the usual sense, whose members are groups satisfying the conditions:

- (a) if $G \in C$ and $G_1 \cong G$ then $G_1 \in C$;
- (b) C contains the trivial group.

We say that a class of groups C is \bar{P} -closed if every group having an ascending series whose factors all lie in C is itself in C . We say that C is L -closed if C contains every group G in which each finite subset of G is contained in some subgroup of G lying in C .

If G is a group we write

$\Delta(G) = \{g \in G : g \text{ has only finitely many } G\text{-conjugates}\}$
and $\Delta^+(G) = \{g \in \Delta(G) : g \text{ has finite order}\}.$

We then have (from Robinson [49], Lemma 4.31 and Theorem 4.32):

Result. For any group G we have:

- (a) $\Delta(G)$ and $\Delta^+(G)$ are characteristic subgroups of G ;
- (b) $\Delta(G)/\Delta^+(G)$ is torsion-free abelian;
- (c) $\Delta^+(G)$ is the union of the finite normal subgroups of G . \square

We call $\Delta(G)$ the FC-subgroup of G (or FC-centre of G).

We say that G is an FC-group when $G = \Delta(G)$.

3. GROUP ALGEBRAS.

We shall, in general, follow Passman [41] and [43] for group algebra notation and terminology.

If K is a field and G is a group we denote the group algebra of G over K by KG or, occasionally, by $K[G]$. If H is a subgroup of G we denote by ωH the right ideal of KG generated by the set $\{1 - h : h \in H\}$. (The reader is warned that many other notations for ωH are used in the references listed at the end of this thesis.) From Lemma 24.3 of [41] we have:

Result. If H is a subgroup of G then

$$\underline{\ell}(\omega H) = \begin{cases} 0 & \dots \text{ if } |H| = \infty \\ KG(\sum_{h \in H} h) & \dots \text{ if } |H| < \infty \end{cases}$$

(where, as usual, $|X|$ means the cardinality of the set X). \square

Connell (in [7], Theorem 8) has characterized prime group algebras as follows:

Result. The group algebra KG is prime if and only if $\Delta^+(G) = 1$ (if and only if $\Delta(G)$ is torsion-free abelian). \square

Group algebras of soluble groups are studied by Zalesskii in [54]. In that paper he proves the following fundamental "intersection theorem":

Result. Let G be a soluble group. There is a characteristic subgroup N of G such that N is an FC-group and, for any nonzero ideal I of KG , we have $I \cap KN \neq 0$. \square

Following Passman [43], p.81, we denote this characteristic subgroup N of G by $3(G)$ and call it the Zaleskiĭ subgroup of G .

CHAPTER 1

COUNTABILITY AND PRIME NONSINGULAR RINGS

SUMMARY

In this chapter we study the maximal right quotient rings of countable, prime, right nonsingular rings. We begin §1 by reviewing Goodearl and Handelman's classification of prime, regular, right self-injective rings and their characterization of those rings whose maximal right quotient rings are simple and right self-injective. Then, in the basic result for this chapter, we show that Goodearl and Handelman's work is greatly simplified when countable rings are involved:

Theorem 5. Suppose R is a countable, prime, right nonsingular ring. Then $Q(R)$, the maximal right quotient ring of R , is either a full linear ring (if R contains uniform right ideals) or a simple, directly infinite ring (if R contains no uniform right ideals).

We show too that this simplification extends to countable dimensional algebras.

In §2 we apply this result to the object of our research, prime nonsingular group algebras, and obtain:

Theorem 13. Suppose G is a group whose conjugacy classes are all countable. If the group algebra KG is prime nonsingular then its maximal right quotient ring is either full linear or simple and directly infinite.

Finally, we give some examples to indicate the diverse types of groups which give prime nonsingular group algebras

(and to prove that we are not looking solely at the trivial group!).

1. COUNTABLE PRIME NONSINGULAR RINGS.

It is well known that if R is a right nonsingular ring then $Q(R)$, the maximal right quotient ring of R , is regular and right self-injective, and that if R is prime then so too is $Q(R)$. Furthermore, Johnson [32] showed that if R , in addition to these two properties, has uniform right ideals then $Q(R)$ is a full linear ring (see Hutchinson [29] for a similar result). Until recently, this was practically all that was known about maximal quotient rings. However, although general maximal quotient rings remain something of an unknown quantity, Goodearl's recent work has taken some of the mystery out of prime, regular, right self-injective rings. Thus in [15] Goodearl proves:

Theorem 1. Suppose Q is a prime, regular, right self-injective ring. Then the (two-sided) ideals of Q are well-ordered by inclusion and each such ideal is of the form

$$H(\aleph) = \{0\} \cup \{q \in Q : qQ \cong E(\aleph(qQ))\}$$

where $\aleph(M)$ means the direct sum of \aleph copies of the module M (\aleph being an infinite cardinal number), and $E(M)$ is the injective envelope of the module M . □

This generalizes Jacobson's well known description of the ideals of a full linear ring (see Jacobson [31], page 93) - a full linear ring being, of course, a prime, regular, right self-injective ring with nonzero socle.

Furthermore, Goodearl and Handelman in [18] classify all prime, regular, right self-injective rings Q showing that Q must be precisely one of the following:

- (a) a full linear ring (this includes both simple Artinian rings - that is, $SP(n)$ rings for some integer $n > 1$, and division rings - and infinite dimensional full linear rings),
- (b) a non-simple ring with zero socle,
- (c) a directly finite, non-Artinian ring (necessarily simple) - that is, an $SP(\infty)$ ring,
- (d) a simple, directly infinite ring - that is, an $SP(1)$ ring which is not a division ring.

Notice that Q , being regular and right self-injective, is a Baer ring and, being prime, is indecomposable (both Renault [47] and Goodearl and Boyle [17] view Q this way). Hence we could classify Q according to the theory of types for Baer rings (see Kaplansky [33]). The two classifications of Q correspond as follows:

Goodearl and HandelmanKaplansky

(a) full linear ring	$\begin{cases} \text{type } I_f, & \text{if } Q \text{ is simple Artinian} \\ \text{type } I_\infty, & \text{if } Q \text{ is infinite dimensional} \end{cases}$
(b) non-simple with zero socle	$\begin{cases} \text{type } II_\infty, & \text{if } H(\aleph_0) \text{ is nonzero} \\ \text{type } III, & \text{if } H(\aleph_0) \text{ is zero} \end{cases}$
(c) directly finite, non-Artinian	type II_f
(d) simple, directly infinite	type III.

However, the theory of types, which is based on the existence of certain idempotents in Q , does not seem so well suited as Goodearl and Handelman's classification to the situation $Q = Q(R)$, where R may well be a ring without nontrivial idempotents.

Goodearl and Handelman go on (in [18]) to characterize the various 'types' of rings R for which $Q(R)$ is one of the simple rings in the above classification (namely, rings of type (c) or (d), or the finite dimensional case of (a)). Since those R for which $Q(R)$ is a full linear ring have been characterized already (see Hutchinson [29] and Johnson [32]) this means that we can, in principle, determine which class $Q(R)$ belongs to for any ring R such that $Q(R)$ is prime, regular and right self-injective. Firstly then, Goodearl and Handelman recall (see [18], Proposition 5.1) the following result from Handelman [22]:

Proposition 2. $Q(R)$ is prime, regular and right self-injective if and only if R is right nonsingular and, for any ideal I of R , $\ell(\ell(I))$ is either 0 or R . \square

(See Johnson [32] for an alternative characterization.) Of course any prime, right nonsingular ring satisfies the conditions of Proposition 2 and we shall in fact restrict our attention to such rings in what follows since, for group algebras KG , we shall see (in Proposition 10 below) that $Q(KG)$ is prime if and only if KG is prime. The class to which $Q(R)$ of Proposition 2 belongs can now be determined by using the next result (see Goodearl and Handelman [18], Theorem 5.3):

Theorem 3. $Q(R)$ is simple and right self-injective if and only if it is prime, regular and right self-injective and at least one of the following holds:

- (i) R has finite right uniform dimension,
- (ii) for any nonzero $r \in R$, there is a large right ideal L of R such that $L \lesssim rR$,
- (iii) if A and B are right ideals of R such that $A \lesssim B$ and A is large then B too is large.

Furthermore, $Q(R)$ is simple and directly infinite (that is, $SP(1)$ but not a division ring) if and only if (ii) holds but (i) does not. $Q(R)$ is directly finite but not Artinian (that is, $SP(\infty)$) if and only if (iii) holds but (i) does not. Finally $Q(R)$ is simple Artinian (that is, $SP(n)$ for some $n > 1$ or a division ring) if and only if (i) holds. \square

It is important to notice that each of the four classes of rings allowed for in Goodearl and Handelman's classification does in fact occur. Examples where Q is of type (a), (c) or (d) are given by them in [18], page 805. For an example where Q is non-simple with zero socle, put $Q = Q(R)$ where $R = T/\text{soc } T$ and T is the full linear ring $\text{End}_K V$ where V is a vector space over the countable field K such that $\dim_K V > 2^{\aleph_0}$ (clearly R , and so Q , is a directly infinite, regular ring with zero socle; Q has a proper ideal since R , and so any large right ideal of R , contains direct sums of $\dim_K V$ nonzero right ideals yet R has nonzero right ideals containing no such direct sum [for example, $\bar{e}R$ is such a right ideal, where \bar{e} is the image in R of an element e of T such that $\dim_K e(V) = \aleph_0$] - see condition (ii) of Theorem 3 above). The examples where Q is of type (b) or (c) are very big and our first result explains why. First, however, we need the following lemma.

Lemma 4. Suppose A, B are nonzero right ideals of the prime, right nonsingular ring R . There are nonzero right ideals $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cong B'$ as R -modules.

Proof. Let $Q = Q(R)$ so that Q is prime and regular. Suppose e, f are idempotents in Q such that eQ and fQ are the injective envelopes of A and B (respectively). As Q is prime we have $0 \neq fQe \cong \text{Hom}_Q(eQ, fQ)$. As Q is regular there are thus nonzero elements $e' \in eQ$ and $f' \in fQ$ and an isomorphism $\psi : e'Q \rightarrow f'Q$. Let $A' = A \cap \psi^{-1}(B \cap f'Q)$ and $B' = \psi(A')$. As A and B are large in eQ and fQ (respectively), A' and B' are nonzero and the lemma is proved. \square

Theorem 5. Suppose R is a countable, prime, right nonsingular ring. Then $Q(R)$ is either a full linear ring (if R has uniform right ideals) or a simple, directly infinite ring (if R has no uniform right ideals).

Proof. The case where R has uniform right ideals is well known (see Faith [9], page 73, or Hutchinson [29], or Johnson [32]) so suppose R has no uniform right ideals. By Theorem 3 it is enough to show that for any nonzero $r \in R$ there is a large right ideal L of R such that $L \lesssim rR$.

Suppose $r \in R$ is nonzero. As R has no uniform right ideals there is an infinite family B_1, B_2, \dots of nonzero independent right ideals contained in rR . Let a_1, a_2, \dots be an enumeration of the nonzero elements of R . We shall construct by induction two families A_1', A_2', \dots and B_1', B_2', \dots of independent right ideals such that, for each integer n ,

- (i) $(A_1' + \dots + A_n') \cap a_n R \neq 0$,
- (ii) $B_n' \subseteq B_n$, and
- (iii) $A_n' \cong B_n'$.

Putting $L = \sum_{n=1}^{\infty} A_n'$ then makes L a large right ideal of R (by (i)) such that $L \cong \sum_{n=1}^{\infty} B_n' \subseteq \sum_{n=1}^{\infty} B_n \subseteq rR$ (by (ii) and (iii)), and this will complete the proof.

By Lemma 4, we can find nonzero right ideals $A_1' \subseteq a_1 R$ and $B_1' \subseteq B_1$ such that $A_1' \cong B_1'$.

Now suppose we have found A_1', \dots, A_K' and B_1', \dots, B_K' satisfying (i), (ii) and (iii) for $1 \leq n \leq K$. If then $(A_1' + \dots + A_K') \cap a_{K+1} R \neq 0$, putting $A_{K+1}' = B_{K+1}' = 0$ completes

the induction step. Otherwise, by Lemma 4, we can find nonzero right ideals $A'_{K+1} \subseteq a_{K+1} R$ and $B'_{K+1} \subseteq B_{K+1}$ such that $A'_{K+1} \cong B'_{K+1}$. In either case A'_1, \dots, A'_{K+1} and B'_1, \dots, B'_{K+1} are still independent and satisfy (i), (ii) and (iii). Hence the induction works and the theorem is proved. \square

Thus those prime, regular, right self-injective rings which are maximal right quotient rings of countable rings R can only be of type (a) or type (d) in the Goodearl-Handelman classification above: type (a) if R has uniform right ideals and type (d) otherwise.

Remark 1. This sort of result, where countable rings are shown to have "desirable" properties, is becoming quite common nowadays. Similar in spirit to Theorem 5, and in fact its original inspiration, is the work in Fisher and Snider [11] and Formanek and Snider [14] on Kaplansky's question: are prime regular rings primitive? In [11], Fisher and Snider show that countable, prime, regular rings are indeed primitive. But here, as in our result above, the countability hypothesis cannot be dropped: in this case Domanov (see Domanov [8] and Passman [46]) has found a prime, regular group algebra (of a fairly ordinary-looking, uncountable, metabelian group) which is not primitive. In another result akin to Theorem 5, Lawrence has shown in [35] that every countable, right self-injective ring is quasi-Frobenius - again, the countability clearly cannot be dropped.

Remark 2. Notice that in the proofs of Lemma 4 and Theorem 5 we do not so much want R , but rather $Q(R)$, to be prime. Hence both these results remain true if we replace the primeness hypothesis on R by the weaker condition in Proposition 2.

Remark 3. A slightly simpler proof of Theorem 5 (avoiding the necessity of an induction argument in the above proof) is available if we use the following result instead of Theorem 3:

Proposition 6. Suppose $Q = Q(R)$ is prime, regular and right self-injective. Then Q is SP(1) if and only if for each nonzero $r \in R$ there is a right ideal $A \subseteq rR$, a large right ideal L' of R and an epimorphism $A \rightarrow L'$.

Proof. Q is SP(1) if and only if each nonzero element of R has a one-element right insulator in Q if and only if for each nonzero $r \in R$ we have $Q \lesssim rQ$ (thus far we have merely copied the proof of Theorem 3). However $Q \lesssim r.Q$ if and only if there is a right ideal \hat{A} of Q with $\hat{A} \subseteq rQ$ and an epimorphism $\psi : \hat{A} \rightarrow Q$. Putting $A = rR \cap \psi^{-1}(R)$ and $L' = \psi(A)$ then completes the proof. \square

Using this, a proof of Theorem 5 would run as follows:

"Suppose r is a nonzero element of R . There is an infinite family B_1, B_2, \dots of nonzero, independent, right ideals contained in rR . Let a_1, a_2, \dots be an enumeration of the nonzero elements of R . For each integer n , there are (by Lemma 4) nonzero right ideals $A'_n \subseteq a_n R$ and $B'_n \subseteq B_n$

such that $A_n' \cong B_n'$. As B_1, B_2, \dots are independent, $\sum_{n=1}^{\infty} B_n'$ is isomorphic to the external direct sum $\bigoplus_{n=1}^{\infty} A_n'$ and so there is an epimorphism from $\sum_{n=1}^{\infty} B_n'$ onto the large right ideal $\sum_{n=1}^{\infty} A_n'$ of R , as required."

However, this method of proof (although equivalent to the original) does not seem to give as much explicit information: the large right ideal L' of Proposition 6 need not, it seems, be embeddable in rR . We shall take advantage of this weaker form when we consider the group algebra of the symmetric group in Chapter 2.

Remark 4. The argument in the previous remark could be extended to uncountable rings R in which each nonzero right ideal contains a family of $|R|$, nonzero, independent, right ideals, and again $Q(R)$ would be simple and directly infinite. However we are not obtaining so much new information this time since merely having one uncountable family of nonzero, independent, right ideals in R forces $Q(R)$ to be directly infinite (if $Q(R)$ were directly finite it would have the dimension function D_S of Theorem 3.17 of Goodearl and Handelman [18] and so could have no uncountable direct sums of nonzero right ideals (see [18], Corollary 3.4)).

Despite Remark 1 we can extend Theorem 5 to some uncountable rings. We need first the following Lemma (see for instance Snider [51], Lemma 3).

Lemma 7. Suppose $F \subseteq K$ are fields and R is an F -algebra. If L is a large right ideal of R then $K \otimes_F L$ is a large right ideal of $K \otimes_F R$. Hence if $K \otimes_F R$ is right nonsingular so is R . □

Theorem 8. Suppose R is a countable dimensional, prime, right nonsingular algebra over the field K . Then $Q(R)$ is either a full linear ring or a simple, directly infinite ring.

Proof. Suppose R has no uniform right ideals. We proceed as in Theorem 5. Suppose r is a nonzero element of R and find nonzero elements b_1, b_2, \dots of rR so that b_1R, b_2R, \dots is an infinite family of independent right ideals. Let x_1, x_2, \dots form a basis for R over K . Then there is a countable subfield F of K and an F -subalgebra S of R such that x_1, x_2, \dots form a basis for S over F , and S contains $\{r, b_1, b_2, \dots\}$. Since $R = KS$ is prime, so is S and, by Lemma 7, S is also right nonsingular. Hence S is a countable, prime, right nonsingular ring and b_1S, b_2S, \dots are independent right ideals contained in rS . By the proof of Theorem 5 (with each $B_n = b_n S$), there is a large right ideal L of S and an S -monomorphism $\psi : L \rightarrow rS$. Then ψ extends to an R -monomorphism $\bar{\psi} : KL \rightarrow rR$ and, by Lemma 7, KL is a large right ideal of R . This completes the proof. □

Remark. This result "explains" the behaviour of the ring $Q(R)$ where R is the ring $\varinjlim_n M_{2n}(F)$ discussed by Goodearl and Handelman in Example (e), page 831 of [18].

Notice, too, that as in Theorem 5 we could weaken the primeness of R to the condition of Proposition 2.

2. PRIME NONSINGULAR GROUP ALGEBRAS

We begin here the principal task of this thesis: seeing how the maximal right quotient rings of prime, right nonsingular group algebras fit into the classification given above for prime, regular, right self-injective rings. Of course we have the following immediate corollary to Theorem 8:

Corollary 9. Suppose G is a countable group such that the group algebra KG is prime and right nonsingular. Then $Q(KG)$ is either a full linear ring or is simple and directly infinite. □

Remark. Nothing is gained this time by replacing the primeness of KG by the usually weaker condition from Proposition 2. In fact we have the following (see Burgess [5], (4.4)).

Proposition 10. For any field K and any group G , $Q(KG)$ is prime if and only if KG is prime.

Proof. If KG is prime then so is $Q(KG)$ (true for any ring). For the converse we first recall Connell's well known result (see Connell [7], Theorem 8) that the group algebra KG is prime if and only if $\Delta^+(G)$, the torsion subgroup of $\Delta(G)$ (the FC-subgroup of G), is trivial. Thus if KG is not prime there is a nontrivial finite normal subgroup N of G . Putting $a = \sum_{g \in N} g$ and $b = |N| - a$, we get nonzero central elements a, b of KG such that $ab = 0$. Hence a, b are central elements of $Q(KG)$ such that $ab = 0$ and so $Q(KG)$ cannot be prime either. □

Thus we are justified in "restricting" our attention to prime, right nonsingular group algebras.

As the primeness of KG depends only on G we shall sometimes find it more convenient to talk of G without mentioning K . In what follows we shall (like Connell [7]) call a group G in which $\Delta^+(G) = 1$ (that is, G contains no nontrivial, finite, normal subgroups) a prime group.

We discuss conditions under which KG is right nonsingular at the end of this chapter.

Since we are trying to see what is special about the maximal right quotient rings of prime, right nonsingular group algebras, Corollary 9 does not really further our cause (after all, the same result holds for any countable dimensional algebra). However, with the help of the following concept we shall show that Corollary 9 also holds for many uncountable groups.

Definition. If every conjugacy class of a group G is countable we call G a CC-group.

This is, of course, analogous to the concept of an FC-group. (Note that for once we are departing from the terminology of Robinson [49].)

Before generalizing Corollary 9, we must prove the following two lemmas:

Lemma 11. Suppose G is a prime CC-group. Every countable subgroup X of G is contained in a countable, prime, normal subgroup \bar{X} of G .

Proof. If N is a normal subgroup of G and if H is a subgroup of G containing N we shall say that the conjugacy classes of N are "fused" in H if each H -conjugacy class contained in N is a complete G -conjugacy class. We shall construct, by induction, a family X_1, X_2, \dots of countable normal subgroups of G such that each $X_n \subseteq X_{n+1}$ and, for each n , the classes of X_n are fused in X_{n+1} . Putting $\bar{X} = \bigcup_{n=1}^{\infty} X_n$ then gives the result since, by construction, the conjugacy classes of \bar{X} are already fused in \bar{X} and since G , being prime, has no finite conjugacy classes consisting of torsion elements.

Let X_1 be the normal closure of X in G (countable because G is a CC-group). Suppose we have found X_1, \dots, X_n . As X_n is countable there is a countable subset Y_n of G such that the classes of X_n are fused in $\langle X_n, Y_n \rangle$. Letting X_{n+1} be the normal closure of $\langle X_n, Y_n \rangle$ completes the construction and so the lemma is proved. \square

Lemma 12. Suppose N is a normal (or subnormal) subgroup of G . Any large right ideal of KN generates a large right ideal of KG . Hence if KG is right nonsingular so is KN .

Proof. (See Burgess [5], 2.5) Let T be a transversal for N in G . Then $KG = \sum_{t \in T} KNt$ and $L.KG = \sum_{t \in T} Lt$, both sums being direct and each Lt being a right KN -submodule of the right KN -module KNt (since N is normal in G). As L is large in KN , each Lt is large in KNt and so $\sum_{t \in T} Lt = L.KG$ is large (as a KN -submodule, and so as a KG -submodule) in KG .

The subnormal case follows by induction on the length of a subnormal chain between N and G . Finally if $a \in Z(KN)$,

$\underline{r}_{KN}(a)$ is a large right ideal of KN and so $\underline{r}_{KG}(a) = \underline{r}_{KN}(a).KG$ is large in KG so that $a \in Z(KG)$. This completes the proof. \square

Theorem 13. Suppose G is a CC-group such that KG is prime and right nonsingular. Then $Q(KG)$ is either a full linear ring or a simple, directly infinite ring.

Proof. Suppose KG has no uniform right ideals and let r be a nonzero element of KG . There are nonzero elements b_1, b_2, \dots of rKG such that b_1KG, b_2KG, \dots is an infinite family of independent right ideals. Applying Lemma 11 to the subgroup of G generated by the supports of r, b_1, b_2, \dots we find a countable, prime, normal subgroup N of G such that r, b_1, b_2, \dots all lie in KN . By Lemma 12, KN is right nonsingular. Since KN is thus a countable-dimensional, prime, right nonsingular algebra, we can apply the proof of Theorem 8 to find a large right ideal L' of KN and a KN -monomorphism $\psi : L' \rightarrow rKN$. By Lemma 12, $L = L'.KG$ is a large right ideal of KG . Since ψ extends to a KG -monomorphism $\bar{\psi} : L \rightarrow rKG$ the proof is complete (by Theorem 3). \square

Remark 1. Since any countable group is clearly a CC-group, Theorem 13 does indeed generalize Corollary 9. Other examples of CC-groups are arbitrary abelian groups, and arbitrary direct products of known CC-groups (in particular, there are uncountable groups for which Theorem 13 holds). There are, however, many examples of groups which are not CC-groups. For instance, uncountable free groups or uncountable simple groups cannot be CC-groups.

Remark 2. Alert readers will have noticed that both extensions of Theorem 5 worked because large right ideals in an appropriate subring generated large right ideals in the ring we were considering. However, large right ideals do not always "go up" in this manner even in group algebras. Formanek [13] gives the following simple example: let G be the free group on two generators a, b and let $H = \langle a \rangle$; then $(1-a)KH$ is a large right ideal of KH but $(1-a)KG$ is not large in KG since $(1-a)KG \cap (1-b)KG = 0$. A different type of example is provided by the restricted symmetric group G on an infinite set X (that is, G is the set of all permutations on X which move only finitely many elements of X): if K has characteristic $p > 0$ and H is a finite subgroup of G of order p then KH has nonzero singular ideal (see Lemma 2.1 of Brown [1]); however, G is a locally finite group and so the radical and singular ideal of KG coincide (see Theorem 3.3 of Brown [1]); since KG is semisimple (by Formanek [12] and Fisher and Snider [11], Theorem 2.7) it must therefore be nonsingular; hence if r is a nonzero element of $Z(KH)$ its right annihilator in KH is a large right ideal of KH which does not generate a large right ideal of KG (see Lemma 12).

In fact Lemma 12 (and various generalizations of it) seems to be the only known case when large right ideals do "go up". This problem obstructs our progress in this thesis in two ways. Clearly, knowing when large right ideals "go up" would be very helpful in generalizing Theorem 13. On the other hand we also need such information when determining which group algebras are in fact nonsingular. An indication

of how difficult this basic problem is is provided by recent work by Passman on the radical in locally finite group algebras (see [43], [44] and [45]); as noted above, this is simultaneously work on the singular ideal. One method of avoiding this obstacle which has had some success in the study of nonsingular group algebras (see Theorem 7 of Snider [51]) is the intersection theorem. Here we have a subgroup H of G such that $KH \cap I \neq 0$ for any nonzero ideal I of KG , and this property is used to reduce the problem to the (hopefully) simpler group algebra KH . We shall find this approach useful for $Q(KG)$ as well, and applications can be found in the main results of Chapters 4 and 5.

Finally in this chapter, to show that we are studying a large enough class of rings to justify the effort, we give some of the more interesting examples of prime, right nonsingular group algebras.

Example 1: Nonsingular Group Algebras. Most of the results in this field can be found in Brown [1], [2] and Snider [51]. We list only those we shall need later on.

The most important source of nonsingular group algebras is Snider's result (see [51], Theorem 4):

Theorem 14. If K is any field of characteristic zero and G is any group then KG is nonsingular. □

Note, incidentally, that because of the natural involution on a group algebra (namely, the function

$\sum_{g \in G} a(g)g \rightarrow \sum_{g \in G} a(g^{-1})g$ a group algebra is right nonsingular if and only if it is left nonsingular. We are thus justified in saying "KG is nonsingular".

More examples of nonsingular group algebras are provided by the following result:

Theorem 15. (See Snider [51], Theorem 7.) Suppose G is a soluble group, and K has characteristic $p > 0$. Then KG is nonsingular if and only if G has no finite subnormal subgroups of order divisible by p . \square

Finally we noted earlier the following:

Theorem 16. If G is a locally finite group then the radical and the singular ideal of KG coincide. In particular if G is the restricted symmetric group on an infinite set then KG is nonsingular for any field K . \square

Example 2: Prime Group Algebras. As mentioned above, KG is prime if and only if $\Delta^+(G) = 1$. Some groups with this property are:

- (a) torsion-free groups,
- (b) infinite simple groups,
- (c) the restricted symmetric group on any infinite set,
- (d) the standard wreath product $A \wr B$ where A is any nontrivial group and B any infinite group (see Lemma 21.5 of Passman [41]),

- (e) any polycyclic or abelian-by-finite group G satisfying $\Delta^+(G) = 1$ (for example, the infinite dihedral group).

Remark. It is easy to pick from these examples instances of prime nonsingular KG for which $Q(KG)$ is full linear and instances where $Q(KG)$ is simple and directly infinite. For example, if G is polycyclic such that $\Delta^+(G) = 1$ then KG is prime and Noetherian (the latter because of a well known theorem due to P. Hall; see Passman [41], page 136) and so $Q(KG)$ is simple Artinian (and so full linear) - see Faith [9], page 84. On the other hand, if G is a noncyclic free group then KG is a non-Ore domain and so $Q(KG)$ is simple and directly infinite (see Example (a), page 826 of Goodearl and Handelman [18]). We shall find more examples (especially of the latter kind) in later chapters.

CHAPTER 2

LOCALLY FINITE GROUP ALGEBRAS

SUMMARY

In this chapter we study the maximal right quotient rings (MRQ rings) of group algebras of locally finite groups. We begin by examining the right ideals of such group algebras and the main tool for this is a dimension function, defined in §1, on the lattice of finitely generated right ideals of a locally finite group algebra. We show that this function imposes certain countability conditions on the group algebra. In §2 the function is used to study minimal right ideals in locally finite group algebras and we obtain:

Theorem 4. Suppose G is a prime, locally finite, nontrivial group. Then for any field K , $\text{soc } KG = 0$.

In §3 we apply the results of the first two sections to MRQ rings of prime, nonsingular, locally finite group algebras. Using Theorem 4 we show that such MRQ rings are never full linear rings and then deduce the main result of this chapter:

Theorem 9. Suppose G is a nontrivial, locally finite group such that KG is prime nonsingular. If G is a CC-group then $Q(KG)$ is simple and directly infinite.

I do not know whether this result remains true if we do not assume that G is a CC-group. We do succeed, however, in showing that Theorem 9 is still true if G is the restricted

symmetric group on any infinite set. We also prove that, even if G in Theorem 9 is not a CC-group, $Q(KG)$ has at most one proper ideal.

Finally in §4 we give further applications of the dimension function from §1. In particular we prove:

Proposition 20. If H is an infinite subgroup of the locally finite group G then ωH is a dense right ideal of KG .

As a corollary we show that when G is locally finite the group algebra KG cannot be its own MRQ ring unless G is finite.

1. A DIMENSION FUNCTION

In this section we introduce a dimension function for locally finite group algebras. Most of the results of this chapter depend on it in one way or another and, since some of these results require neither the primeness nor the non-singularity of the group algebra concerned, we define the dimension function for arbitrary locally finite group algebras. [I later discovered that this function had been defined by Goursaud and Jeremy (see [19], Proposition 2.9) in the special case of regular group algebras (that is, group algebras KG where K has characteristic $p \geq 0$ and G is locally finite with no elements of order p).]

Suppose G is a locally finite group. We shall denote by $L(KG)$ the lattice of finitely generated right ideals of the group algebra KG . We define the function $d: L(KG) \rightarrow [0,1]$ as follows:

if $A = a_1 KG + \dots + a_n KG \in L(KG)$ choose a finite subgroup H of G containing the supports of a_1, \dots, a_n and set

$$d(A) = |H|^{-1} \dim_K(a_1 KH + \dots + a_n KH).$$

Proposition 1. If G is a locally finite group then $d : L(KG) \rightarrow [0, 1]$ is well-defined, and if $A, B \in L(KG)$ then we have:

- (i) $d(A) \geq 0$ with equality if and only if $A = 0$,
- (ii) $d(B) \leq 1$ with equality if and only if $B = KG$,
- (iii) $d(A+B) \leq d(A) + d(B)$ with equality if $A \cap B = 0$,
- (iv) $A \lesssim B$ implies $d(A) \leq d(B)$.

Proof. We show first that $d(A)$ is independent of the choice of H . It is enough to consider a finite subgroup H_1 of G containing H . Let T be a transversal for H in H_1 (via right cosets). Then

$$\sum_{i=1}^n a_i KH_1 = \sum_{t \in T} \left(\sum_{i=1}^n a_i KH \right) t$$

and the sum over T is a direct sum of isomorphic K -modules. Hence, since $|H_1| = |H| \cdot |T|$, we have

$$\begin{aligned} |H_1|^{-1} \dim_K \left(\sum_{i=1}^n a_i KH_1 \right) &= |H_1|^{-1} |T| \dim_K \left(\sum_{i=1}^n a_i KH \right) \\ &= |H|^{-1} \dim_K \left(\sum_{i=1}^n a_i KH \right) \end{aligned}$$

as required. That $d(A)$ is independent of the choice of a_1, \dots, a_n now follows easily. The properties (i) to (iv) are consequences of the corresponding properties of \dim_K . \square

There is, of course, a corresponding dimension function on the lattice of finitely generated left ideals of a locally finite group algebra.

The following Proposition shows that our dimension function imposes certain countability conditions on locally finite group algebras. It was first proved by Goursaud and Jeremy (see Corollaire 2.10 of [19]) in the special case when KG is regular.

Proposition 2. Suppose G is locally finite. Then:

- (i) KG has no uncountable families of nonzero independent right ideals;
- (ii) every right ideal of KG is an essential extension of a countably generated right ideal;
- (iii) KG has no infinite families of nonzero, independent, pairwise isomorphic right ideals.

Proof. We shall prove (i) by imitating the proof of Lemma 13 of Goodearl [16]. Suppose X is a family of nonzero independent right ideals of KG . We may assume that the members of X are finitely generated. For each positive integer n , let $X_n = \{A \in X : d(A) > 1/n\}$ so that, by Proposition 1(i), we have $X = \cup X_n$. If some $|X_n| \geq n$ we can choose distinct A_1, \dots, A_n in X_n which, being independent, give

$$d(A_1 + \dots + A_n) = d(A_1) + \dots + d(A_n) > 1$$

(by Proposition 1(iii)). Since this contradicts Proposition 1(ii), we must have $|X_n| < n$ for all n and so $X = \cup X_n$ is

countable, as required. Finally (ii) follows from (i), while a similar argument gives (iii). \square

Remark. Suppose G is a locally finite group such that KG is nonsingular. One consequence of Proposition 2(ii) is that any KG -homomorphism $\psi : A \rightarrow KG$ (where A is a (large) right ideal of KG) is determined by its action on a countably generated right ideal A' large in A .

2. SOCLE IN PRIME LOCALLY FINITE GROUP ALGEBRAS

As part of the classification of MRQ rings of prime, nonsingular group algebras we have to determine when such group algebras have uniform right ideals (that is, when the MRQ ring is full linear). This is especially important when we are dealing with CC-groups since then, as Theorem 1.13 shows, the presence or absence of uniform right ideals is a complete test of whether the MRQ ring is full linear or simple and directly infinite. Since any minimal right ideal in a ring is also uniform, a useful preliminary step is to find which group algebras have nonzero socle. In this section we show that prime group algebras of (nontrivial) locally finite groups always have zero socle.

The socle of a group algebra KG , where G is an infinite locally finite group, was first studied by Müller in [40]. He showed that in many cases this socle had to be zero. For instance, if K is an algebraically closed field, we always have $\text{soc } KG = 0$ (see Satz 2.2 of [40]). Another result, not dependent on the field K this time, is the following (see Satz 2.3 of [40]):

Proposition 3. Suppose G is a locally finite group such that, for any finite subgroup U of G , there is a non-trivial finite subgroup V of G normalized by U with $U \cap V = 1$. Then KG has zero socle for any field K . \square

However, there are some cases where KG has nonzero socle: for example, when K is the field of rational numbers and G is the Prüfer group Z_{p^∞} (see Satz 2.5 of Müller [40]).

I do not know whether all prime locally finite groups satisfy the condition of Proposition 3, but the following simple application of the dimension function from §1 avoids such problems.

Theorem 4. Suppose G is a prime, locally finite, non-trivial group. For any field K we have $\text{soc } KG = 0$.

Proof. If H is a finite subgroup of G then $d((\sum_{h \in H} h)KG) = |H|^{-1}$ since $\sum_{h \in H} h$ generates $(\sum_{h \in H} h)K$ over KH . But G , being a nontrivial prime group, is infinite. Hence we have $\inf\{d(A) : A \neq 0\} = 0$.

Suppose, however, that B is a minimal nonzero right ideal of KG and let A be any nonzero finitely generated right ideal of KG . As KG is a prime ring we have $B \lesssim A$ and so, by Proposition 1(iv), $d(B) \leq d(A)$. Hence, by Proposition 1(i), we have the contradiction $0 < d(B) = \inf\{d(A) : A \neq 0\} = 0$. Thus $\text{soc } KG = 0$, as required. \square

Hartley and Richardson later characterized those fields K and locally finite groups G for which $\text{soc } KG \neq 0$ (see [27]).

and [48]). In particular, Richardson shows (in Theorem 3.1 of [48]) the following result:

Theorem 5. Suppose G is a locally finite group such that $\text{soc } KG \neq 0$. Then G is a Cernikov group (that is, there is an abelian subgroup A of finite index in G and A is a finite direct product of quasi-cyclic groups). \square

Of course, our Theorem 4 now follows from Theorem 5 but the proof of this latter result uses some deep results from group theory and our original elementary proof of Theorem 4 seems preferable in the prime case.

3. MAXIMAL RIGHT QUOTIENT RINGS

Some MRQ rings of locally finite group algebras have already been studied by Goursaud and Valette in [20]. In fact, working in essence with Passman's generalized polynomial identities (see Passman [42]), they prove the following result:

Proposition 6. Suppose that K is a field of characteristic $p \geq 0$ and that, if $p = 0$, K contains all the roots of unity. If KG is regular then the following conditions are equivalent:

1. G has an abelian subgroup of finite index.
2. $Q(KG)$ is of type I.
3. $Q(KG)$ is of type I finite, bounded. \square

In this section we shall look at $Q(KG)$ when G is locally finite and KG is prime nonsingular. Since a prime, locally finite, nontrivial group G cannot be abelian-by-finite (if A were an abelian subgroup of finite index, we would have $A \subseteq \Delta(G)$ and yet $\Delta(G) = 1$ since G is prime and locally finite) we are, so to speak, examining the opposite side of the coin to that studied by Goursaud and Valette.

Notice that Proposition 6 implies that, subject to the restrictions there on K and G , $Q(KG)$ is not a full linear ring. We shall remove these restrictions on K and G but firstly we need the following definition and lemma from Fisher and Snider [11].

Definition. Let R be a ring with identity 1 . We say that R is locally Artinian if every finite subset of R is contained in an Artinian subring of R which contains 1 .

Although this differs slightly from the notion in [11] with the same name, the same proof as is used in [11] for Lemma 2.1 still gives the following lemma:

Lemma 7. If R is a semisimple, locally Artinian ring then every nonzero right ideal of R contains a nonzero idempotent. □

Proposition 8. Suppose G is a nontrivial locally finite group. Then $Q(KG)$ is never a full linear ring.

Proof. Suppose $Q(KG)$ is a full linear ring. Then KG is prime, nonsingular and has uniform right ideals. As G is locally finite, KG is locally Artinian and as KG is nonsingular, it is semisimple (by Theorem 1.16). It follows from Lemma 7 that every uniform right ideal of KG is a minimal right ideal and so $\text{soc } KG \neq 0$. This contradiction of Theorem 4 completes the proof. \square

Because of Theorem 1.13 our main result is an immediate corollary to Proposition 8:

Theorem 9. Suppose G is a nontrivial, locally finite, CC-group such that KG is prime nonsingular. Then $Q(KG)$ is simple and directly infinite. \square

Thus, of the four original possibilities allowed for in the Goodearl-Handelman classification discussed in Chapter 1, only one (namely (d)) occurs as $Q(KG)$ where KG is prime nonsingular and G is a nontrivial, locally finite, CC-group. Even if G were not a CC-group $Q(KG)$ could never be of type (a).

Theorem 9 provides an interesting contrast to Goursaud and Valette's work in [20] (see Proposition 6 above). They show that, subject to the restrictions on K and G mentioned in Proposition 6 above, the type I part of $Q(KG)$ is always directly finite (see the Remarques after Theoreme 1.3 of [20]). On the other hand there is an interesting parallel with Fisher and Snider's result (see [11]) that if KG is prime nonsingular and G is a countable, locally finite group then KG is primitive.

As we remarked in Chapter 1 (see Remark 2 after Theorem 1.13) we could extend Theorem 9 to more general locally finite groups if we knew when large right ideals in a suitable subring generated large right ideals in our group algebra. We shall consider now the special case of the symmetric group algebra where we can in fact find enough such large right ideals 'going up'.

If X is any infinite set we shall denote by S_X the group of all permutations of X which move only finitely many elements of X (that is, S_X is the restricted symmetric group on X). Then S_X is a prime, locally finite group and, as we noted in Theorem 1.16, KS_X is nonsingular for any field K . However if X is uncountable S_X is clearly not a CC-group.

Let us call a subgroup H of S_X a squashed subgroup of S_X if there is a subset W of X such that $H \subseteq S_W$ and $X \setminus W$ is infinite. Using this notion we can identify some large right ideals of KS_X which will 'go up' when S_X is embedded in some S_Y (Y being a set containing X).

Lemma 10. Suppose $\sum_{n=1}^{\infty} a_n KS_X$ is a large right ideal of KS_X and $\langle \text{supp } a_n : 1 \leq n < \infty \rangle$ is a squashed subgroup of S_X . Then for any set Y containing X , $\sum_{n=1}^{\infty} a_n KS_Y$ is a large right ideal of KS_Y .

Proof. It is enough to consider the case where $Y \setminus X$ is finite. Then, since $\langle \text{supp } a_n : 1 \leq n < \infty \rangle$ is a squashed subgroup of S_X , there is a group isomorphism $S_X \rightarrow S_Y$ which fixes $\langle \text{supp } a_n : 1 \leq n < \infty \rangle$. This isomorphism induces a ring isomorphism $KS_X \rightarrow KS_Y$ which fixes each of the elements

a_n ($1 \leq n < \infty$). Hence $\sum_{n=1}^{\infty} a_n KS_Y$, being the image of the large right ideal $\sum_{n=1}^{\infty} a_n KS_X$ of KS_X , is a large right ideal of KS_Y , as required. \square

Remark. One consequence of Lemma 10 is that when X is uncountable, every large right ideal of KS_X generates a large right ideal of KS_Y . Indeed, suppose L is large in KS_X . As S_X is locally finite, Proposition 2(ii) gives us countably many elements a_1, a_2, \dots in L with $\sum_{n=1}^{\infty} a_n KS_X$ large in L (and so in KS_X). As X is uncountable, $\langle \text{supp } a_n : 1 \leq n < \infty \rangle$ must be squashed in S_X and so, by Lemma 10, $\sum_{n=1}^{\infty} a_n KS_Y$ is large in KS_Y . Hence so is $L.KS_Y$, as claimed. However, this fact does not help us reduce our problem back to the countable case.

Roughly speaking, Lemma 10 tells us that the large right ideal L of KS_X will 'survive' the embedding $KS_X \rightarrow KS_Y$ provided it has already survived an embedding $KS_W \rightarrow KS_X$ such that $X \setminus W$ is infinite. Our next lemma shows us how to construct such large right ideals in the countable case.

Lemma 11. Suppose F is a countable field and X is a countably infinite set. Let r be a nonzero element of FS_X . Then there is a large right ideal $L = \sum_{n=1}^{\infty} c_n FS_X$ of FS_X such that $\langle \text{supp } c_n : 1 \leq n < \infty \rangle$ is squashed in S_X , a right ideal $A \subseteq r FS_X$ and an FS_X -epimorphism $A \rightarrow L$.

Proof. We can choose a subset W of X such that $\text{supp } r \subseteq S_W$ and yet $|X \setminus W| = |X| = |W|$. Then FS_W is prime

nonsingular (as W is infinite) and so has no uniform right ideals (by Proposition 8). Thus there are elements b_1, b_2, \dots in rFS_W such that $\{b_n FS_W : 1 \leq n < \infty\}$ is an infinite family of nonzero independent right ideals of FS_W . Let a_1, a_2, \dots be an enumeration of the elements of FS_X . For each positive integer n , there is some $w_n \in S_W$ such that $b_n^{w_n}$ is 'disjoint' from a_n (that is, no element of X is moved by both $\text{supp } b_n^{w_n}$ and $\text{supp } a_n$ - possible because W is infinite). Thus we have $0 \neq a_n b_n^{w_n} = b_n^{w_n} a_n$. Putting $c_n = b_n^{w_n}$ then gives $c_n FS_X \cap a_n FS_X \neq 0$ and $c_n FS_W \cong b_n FS_W$ for each n . Hence $L = \sum_{n=1}^{\infty} c_n FS_X$ is large in FS_X and $\langle \text{supp } c_n : 1 \leq n < \infty \rangle$ is squashed in S_X . Finally we have a right ideal $A = \sum_{n=1}^{\infty} b_n FS_X \subseteq rFS_X$ and an epimorphism

$$A = \sum_{n=1}^{\infty} b_n FS_X \rightarrow \sum_{n=1}^{\infty} c_n FS_X = L$$

since $b_1 FS_X, b_2 FS_X, \dots$ are independent. □

Theorem 12. Suppose K is a field and Y is any infinite set. Then $Q(KS_Y)$ is simple and directly infinite.

Proof. As might be expected from Lemma 11, we shall use the criterion from Proposition 1.6 rather than that in Theorem 1.3 which we have used previously. Suppose $0 \neq r \in KS_Y$. We just need to find a large right ideal L of KS_Y , a right ideal $A \subseteq rKS_Y$ and a KS_Y -epimorphism $A \rightarrow L$.

Choose a countable subfield F of K and a countably infinite subset X of Y such that $r \in \text{FS}_X$. By Lemma 11, there is a large right ideal $L' = \sum_{n=1}^{\infty} c_n \text{FS}_X$ of FS_X such that $\langle \text{supp } c_n : 1 \leq n < \infty \rangle$ is a squashed subgroup of S_X , a right ideal $A' \subseteq r\text{FS}_X$ and an FS_X -epimorphism $\psi : A' \rightarrow L'$. Clearly ψ induces a KS_Y -epimorphism from $A = A' \cdot \text{KS}_Y$ onto $L = L' \cdot \text{KS}_Y$. By Lemma 10, $L' \cdot \text{FS}_Y$ is large in FS_Y since $\langle \text{supp } c_n : 1 \leq n < \infty \rangle$ is a squashed subgroup of S_X . Hence by Lemma 1.7, $L = (L' \cdot \text{FS}_Y) \cdot K$ is large in KS_Y and the proof is complete. \square

Remark. When K has zero characteristic in Theorem 12, we can use the regularity of KS_Y to retrieve from the epimorphism $A \rightarrow \sum_{n=1}^{\infty} c_n \text{KS}_Y$ a monomorphism $\sum_{n=1}^{\infty} c_n \text{KS}_Y \rightarrow A \subseteq r\text{KS}_Y$ (basically because $\sum_{n=1}^{\infty} c_n \text{KS}_Y$ is projective). Thus in this case we could use Theorem 1.3 again instead of Proposition 1.6. However when K has positive characteristic it is not clear whether such a construction is possible. Certainly Proposition 1.6 is easier to apply here (see Remark 3 after Theorem 1.5).

The argument used here to handle the restricted symmetric groups does not seem to work for other groups: the technique for manufacturing suitable large right ideals appears to rely quite heavily on the structure of S_X . However there is a small gleam of hope: Lemma 10 shows that there are other ways to make large right ideals behave well besides requiring the presence of 'enough' normal subgroups.

Despite this, I do not know whether Theorem 9 remains true for all nontrivial locally finite groups. We shall see

in Chapter 4 that it certainly holds for arbitrary soluble locally finite groups but in the general case (which includes such diverse groups as uncountable, simple, locally finite groups and uncountable, locally nilpotent, prime, p -groups) the most we can say so far is the following result: at worst, $Q(KG)$ is only one step away from being simple.

Proposition 13. Suppose G is a nontrivial locally finite group such that KG is prime nonsingular. Then $Q(KG)$ has at most one proper ideal and, in the notation of Theorem 1.1, $Q(KG) = H(N_1)$.

Proof. By Proposition 2(i), KG has no uncountable direct sums of nonzero right ideals. Hence neither has $Q(KG)$. By the definition of $H(N_1)$ (see Theorem 1.1), we must have $Q(KG) = H(N_1)$ and so the only possible proper ideal of $Q(KG)$ is $H(N_0)$. □

Example. Many prime, nonsingular, locally finite group algebras KG are included in the examples at the end of Chapter 1. To make KG nonsingular we could, for instance, choose any field K with $\text{char } K = p \geq 0$ where, if $p > 0$, G has no elements of order p . Examples where G is prime are:

- (a) infinite, simple, locally finite groups G (for instance, G could be $PSL(n, F)$ where $n \geq 2$ and F is an infinite locally finite field),
- (b) the standard wreath product $G = A \wr B$ where A and B are nontrivial locally finite groups and B is infinite (in particular, every locally finite group can be embedded in a locally finite prime group).

4. MORE APPLICATIONS OF THE DIMENSION FUNCTION

In this section we depart momentarily from our main theme of prime nonsingular group algebras to consider more applications of the dimension function d in §1.

Recall first that if H is an infinite normal subgroup of an arbitrary group G then ωH is an ideal of KG such that $\underline{\ell}(\omega H) = 0$, and so ωH is large (indeed, dense) as a right ideal of KG . Using Lemma 1.12 it is easy to see that ωH is still large if H is only subnormal in G . When G is locally finite, however, we can completely remove the conditions on H .

Proposition 14. If H is any infinite subgroup of the locally finite group G then ωH is a large right ideal of KG .

Proof. If \bar{H} is any finite subgroup of H we have $d(\omega \bar{H}) = 1 - |\bar{H}|^{-1}$. Since H is infinite, it follows that $\sup\{d(A) : A \subseteq \omega H\} = 1$. Hence ωH is large (for if B is a nonzero finitely generated right ideal of KG there is some $A \subseteq \omega H$ with $d(A) > 1 - d(B)$ so that if $A \cap B = 0$ we would have the contradiction $1 < d(A) + d(B) = d(A + B) \leq 1$). \square

Remark 1. Although G does not have to be locally finite for Proposition 14 to remain true (for example, G could be nilpotent instead: then every subgroup of G is subnormal in G), some condition on G is necessary. For instance, consider Formanek's example from [13] (see Remark 2 after Theorem 1.13).

Remark 2. Since, in Proposition 14, the augmentation ideal of KH was already known to be a large right ideal of KH , this result gives another example of a large right ideal which 'goes up'. However ωH is a special case: there can still be some large right ideals of KH which do not survive the embedding $KH \subseteq KG$ even when H is infinite. For instance, let G be the restricted symmetric group on an infinite set, H an infinite elementary abelian p -subgroup of G and suppose that $\text{char } K = p > 0$. Then we know that $Z(KG) = 0$ (by Theorem 1.16) but that $Z(KH) \neq 0$ (by Lemma 21.5(i) of Passman [41] and by Theorem 1.16). Hence there are large right ideals of KH (namely, annihilators of nonzero elements of $Z(KH)$) which do not generate large right ideals of KG . Notice too that the right ideals ωH are not suitable (the way those discussed in Lemma 10 were) for showing $Q(KG)$ is simple and directly infinite: if $r \in KG$ and there is a right ideal $A \subseteq rKG$ with an epimorphism $A \rightarrow \omega H$ then the fact that $\sup \{d(B) : B \subseteq \omega H\} = 1$ ensures that $d(rKG) = 1$ and so rKG cannot be a proper right ideal.

Richardson shows in [48] that Proposition 14 can be used to study the socle of a locally finite group algebra. The following result has been extracted from his proof of Theorem 3.1 of [48].

Lemma 15. Suppose G is a locally finite group such that $\text{socle } KG \neq 0$. Then every residually finite subgroup of G is finite.

Proof. Suppose H is an infinite, residually finite subgroup of G and let X be the set of all subgroups of H of finite index in H . Then X is closed under finite intersections and, as H is residually finite, $\bigcap_{\bar{H} \in X} \bar{H} = 1$. By Lemma 2.1 of Wallace [52] it follows that $\bigcap \{\omega \bar{H} : H \in X\} = 0$. But if $\bar{H} \in X$, $\omega \bar{H}$ is a large right ideal of KG (by Proposition 14) and so contains $\text{soc } KG$. Hence $\text{soc } KG = 0$ which is a contradiction. \square

Richardson uses this result to deduce Theorem 5 above. We shall use it to generalise Theorem 4 to a form required in Chapter 3. Of course, we could deduce the following result from Theorem 5 itself, but we prefer a more elementary approach which avoids the deep group theoretic results used by Richardson.

Proposition 16. Suppose G is a nontrivial, locally finite group containing no finite, nontrivial, characteristic subgroups. For any field K , $\text{soc } KG = 0$.

Proof. If G is prime the result follows from Theorem 4. Hence, by hypothesis, $\Delta(G)$ is an infinite subgroup of G . Since $\Delta(G)$ is a normal subgroup of G , $\text{soc } KG \neq 0$ implies that $\text{soc } K[\Delta(G)] \neq 0$ (by the straightforward Lemma 2.1(d) of [48]). As $\Delta(G)$ cannot have finite, nontrivial, characteristic subgroups either, we may suppose that $G = \Delta(G)$ and $\text{soc } KG \neq 0$, and derive a contradiction. If G had trivial centre, G would be a residually finite group (since G is an FC-group) and we would have a contradiction of Lemma 15. So suppose

$Z \neq 1$ is the centre of G . By Proposition 3, there is a finite subgroup U of G such that $U \cap V \neq 1$ for any nontrivial, finite subgroup V of G normalized by U . In particular, U contains all the elements of Z of prime order. Hence, as U is finite, G has a finite, nontrivial, characteristic subgroup and this contradiction completes the proof. \square

We shall now show that if H is an infinite subgroup of the locally finite group G then ωH is in fact a dense right ideal of KG . For this strengthening of Proposition 14 we need the following Lemmas.

Lemma 17. Let G be a locally finite group and A a finitely generated right ideal of KG . Then any KG -homomorphism $f : A \rightarrow KG$ extends to a KG -homomorphism $\bar{f} : KG \rightarrow KG$.

Proof. This follows from Theorems 1 and 3 of Colby [6] but the direct approach also works: Suppose a_1, \dots, a_n generate A and choose a finite subgroup H of G containing $\bigcup_1^n \text{supp } a_i$ and $\bigcup_1^n \text{supp } f(a_i)$. Then f restricts to a KH -homomorphism g from $A' = a_1 KH + \dots + a_n KH$ to KH and so, as KH is self-injective, g extends to a KH -homomorphism $\bar{g} : KH \rightarrow KH$. Finally \bar{g} induces a KG -homomorphism $\bar{f} : KG \rightarrow KG$ which is easily seen to extend f . \square

Lemma 18. Suppose G is locally finite and A_1, \dots, A_n are finitely generated right ideals of KG . Then

$$\underline{\ell}(A_1 \cap A_2 \cap \dots \cap A_n) = \underline{\ell}(A_1) + \dots + \underline{\ell}(A_n).$$

Proof. The case $n = 2$ follows from Lemma 17 because of Theorem 1(ii) of Ikeda and Nakayama [30]. The general case follows by induction because the intersection of any pair of finitely generated right ideals of KG is again finitely generated (since KG is coherent by Theorems 2 and 3 of Colby [6] although once again a direct proof is available). □

Lemma 19. Suppose H_1, \dots, H_n are infinite subgroups of the locally finite group G . Then

$$\underline{\ell}\left(\bigcap_{i=1}^n \omega H_i\right) = 0.$$

Proof. Denote by d the dimension function on the finitely generated left ideals of KG (see Proposition 1).

Suppose $0 \neq x \in \underline{\ell}\left(\bigcap_{i=1}^n \omega H_i\right)$ so that $d(KGx) > 0$. For each i ($1 \leq i \leq n$) choose a finite subgroup \bar{H}_i of H_i such that $|\bar{H}_i| > n/d(KGx)$. Since $\bigcap_{i=1}^n \omega \bar{H}_i \subseteq \bigcap_{i=1}^n \omega H_i$ we have $x \in \underline{\ell}\left(\bigcap_{i=1}^n \omega \bar{H}_i\right)$ (which is finitely generated since KG is coherent) and so

$$\begin{aligned} d(KGx) &\leq d\left(\underline{\ell}\left(\bigcap_{i=1}^n \omega \bar{H}_i\right)\right) \\ &= d\left(\sum_{i=1}^n \underline{\ell}(\omega \bar{H}_i)\right) && \text{(by Lemma 18)} \\ &\leq \sum_{i=1}^n d(\underline{\ell}(\omega \bar{H}_i)) && \text{(by Proposition 1)} \\ &= \sum_{i=1}^n |\bar{H}_i|^{-1} && \text{(since if } H \text{ is any finite} \end{aligned}$$

subgroup of G , $\underline{\ell}(\omega H) = KG(\sum_{h \in H} h)$). Hence, by our choice of $\bar{H}_1, \dots, \bar{H}_n$ we have

$$d(KGx) < \sum_{i=1}^n n^{-1} d(KGx) = d(KGx)$$

which is a contradiction. □

Proposition 20. Suppose H is an infinite subgroup of the locally finite group G . Then ωH is a dense right ideal of KG .

Proof. We have to show that, for any $a \in KG$, $\underline{\ell}(\omega H : a) = 0$ where $(\omega H : a) = \{b \in KG : ab \in \omega H\}$. Say $a = a_1 g_1 + \dots + a_n g_n \in KG$ where each $g_i \in G$ and each $a_i \in K$, $a_i \neq 0$. Then we have $(\omega H : a) \supseteq \bigcap_{i=1}^n (\omega H : a_i g_i) = \bigcap_{i=1}^n (\omega H : g_i) = \bigcap_{i=1}^n (g_i^{-1}(\omega H)g_i) = \bigcap_{i=1}^n \omega(g_i^{-1}Hg_i)$ and since each $g_i^{-1}Hg_i$ is an infinite subgroup of G we have $\underline{\ell}(\omega H : a) \subseteq \underline{\ell}(\bigcap_{i=1}^n \omega(g_i^{-1}Hg_i)) = 0$ by Lemma 19. This completes the proof. □

If, in Proposition 20, we take H to be countable then we see that ωG is a rational extension of a countably generated right ideal of KG . It would be interesting to know whether this is true for any right ideal of KG and not just ωG (we already know from Proposition 2 that each right ideal is an essential extension of a countably generated right ideal).

As an application of Proposition 20 we shall show that if G is locally finite then KG is rationally complete (that is, is its own maximal right quotient ring) only if G is finite. Lawrence and Loudon have studied rationally complete group algebras in [36] and they showed that if G is an arbitrary countable group such that KG is rationally complete then G is finite.

Proposition 21. Suppose G is a locally finite group such that KG is rationally complete. Then G is finite.

Proof. Suppose G is infinite and let H be a countably infinite subgroup of G . By Proposition 20, ωH is a dense right ideal. To complete the proof it is thus enough to find a KG -homomorphism $\bar{f} : \omega H \rightarrow KG$ which cannot be given by left multiplication by an element of KG . Thus it is enough to find a KH -homomorphism $f : \bar{\omega} H \rightarrow KH$ (where $\bar{\omega} H$ denotes, temporarily, the augmentation ideal of KH) such that f cannot be given by left multiplication by an element of KH (such an f would induce a KG -homomorphism $\bar{f} : \omega H \rightarrow KG$ and if \bar{f} were given by left multiplication from KG , its restriction to $\bar{\omega} H$ could be given by left multiplication from KH). Hence we may assume that $G = H$ is countable and so we simply copy Farkas' proof that if KG is self-injective then G is finite (see Farkas [10]):

We can find a strictly increasing chain $G_1 \subset G_2 \subset \dots$ of finite subgroups of G whose union is G . We define a sequence a_1, a_2, \dots of elements of KG by setting $a_1 = 1$ and requiring that $a_{n+1} = a_n + \sum_{g \in G_n} g$ for each n . Then we can define $f : \omega G \rightarrow KG$ by setting $f(x) = a_n x$ when $x \in \omega G_n$.

Then, as in Farkas [10], we can show that f cannot be given by left multiplication by any element of KG . This completes the proof. □

I do not know whether Proposition 21 is true for arbitrary groups G . Suppose, however, that G is any infinite group for which KG is rationally complete. By Proposition 21

and by Proposition 3 of Lawrence and Loudon [36], the unique, maximal, locally finite, normal subgroup N of G must be finite. Hence by Proposition 8 of [36], we can pass to the group G/N , which has no nontrivial, locally finite, normal subgroups, while still having $K[G/N]$ rationally complete. Now Handelman and Lawrence have conjectured in [23] that the group algebra of a group containing no nontrivial, locally finite, normal subgroups is a strongly prime ring, and it is known that any strongly prime ring is nonsingular. Since a nonsingular, rationally complete ring is self-injective, this would suggest that $K[G/N]$ is self-injective and so that G/N is finite. This contradiction suggests that Proposition 21 holds for all groups G .

CHAPTER 3

UNIFORM RIGHT IDEALS

SUMMARY

Suppose KG is prime, nonsingular and has uniform right ideals. Then $Q(KG)$ is either simple Artinian or an infinite dimensional full linear ring. In this chapter we show that for many classes of groups only the first alternative ever occurs.

In §1 we apply Formanek's result (from [13]) on MRQ rings of group algebras of normal subgroups to deduce:

Theorem 5. If KG is prime, nonsingular and has uniform right ideals then G contains no nontrivial, locally finite, normal subgroups.

Handelman and Lawrence have conjectured that this condition on G implies that KG is strongly prime, and this suggests that KG in Theorem 5 has a simple Artinian MRQ ring. In §§2, 3 we take a closer look at this suggestion by considering the following situation:

(*) ... KG is prime, nonsingular and has uniform right ideals, N is a normal subgroup of G and $Q(KN)$ is simple Artinian.

We show that if G/N in (*) is residually finite (§2) or locally finite (§3) then $Q(KG)$ is simple Artinian too. The locally finite case is handled using a dimension function

like that in Chapter 2, and this function is also used (in §3) to extend some of Lawrence and Loudon's work (in [36]) on rationally complete group rings. We apply these results (in §4) to deduce the following (see Corollary 11, Theorem 19 and Proposition 21):

Result. Suppose KG is prime, nonsingular and has uniform right ideals. If either

- (a) G is residually finite, or
- (b) G is soluble, or
- (c) G is linear and $\text{char } K = 0$,

then $Q(KG)$ is simple Artinian.

Finally we discuss whether this result holds for all groups G and review the known results about the existence of uniform right ideals in group algebras.

1. NORMAL SUBGROUPS AND UNIFORM RIGHT IDEALS.

Suppose KG is prime, nonsingular and has uniform right ideals, and suppose N is a normal subgroup of G . We consider here the consequences of our assumptions for the structure of N . The main tool used in this investigation is the following result (proved by Formanek in [13]).

Lemma 1. Suppose N is a subnormal subgroup of the group G . There is a natural embedding of $Q(KN)$ into $Q(KG)$ and this allows us to make the following identification:

$$Q(KN) = \{q \in Q(KG) : \{a \in KN : qa \in KN\} \text{ is a dense right ideal of } KN\}.$$

□

We note that all that is needed in the proof of Lemma 1 is that dense right ideals of KN generate dense right ideals of KG (of course, if KG is non-singular this is the same as requiring that large right ideals 'go up' from KN to KG). Thus, for instance, Lemma 1 is still true if N is merely ascendant in G . However Formanek gives a simple example in [13] to show that $Q(KN)$ need not be embeddable in $Q(KG)$ if dense right ideals of KN do not 'go up' to KG .

Whenever dense right ideals of KN do generate dense right ideals of KG we shall, without further comment, assume the above identification has been made.

Remark. An immediate consequence of Lemma 1 is that if N is a subgroup of G for which the embedding of Lemma 1 exists, and if the ring $Q(KN)$ is directly infinite, then $Q(KG)$ is also directly infinite. In particular, if G is the complete symmetric group on any infinite set X (that is, the group of all permutations of X) then $Q(KG)$ is directly infinite since the restricted symmetric group S_X is a normal subgroup of G and since $Q(KS_X)$ is directly infinite (by Theorem 2.12).

The embedding of Lemma 1 is especially useful when N is normal in G as we shall see in the next result (also proved by Loudon in Corollary 12 of [39]). Firstly we need a definition (see Passman [41], page 65).

Definition. Let R be a subring of the ring S containing the identity of S . We say that S has a normalizing basis B

over R if S is a free left R -module with basis B and, for each $b \in B$, there is an automorphism σ_b of R such that, for all $r \in R$, $br = \sigma_b(r)b$. (We do not assume that B is finite.)

Lemma 2. Suppose N is a normal subgroup of G and let T be a transversal for N in G . Then:

- (a) for any $g \in G$ the function $q \rightarrow gqg^{-1}$ is an automorphism of the ring $Q(KN)$;
- (b) the subring of $Q(KG)$ generated by $Q(KN)$ and KG has normalizing basis T over $Q(KN)$ and is a right quotient ring of KG .

Proof. Since N is normal in G , (a) follows from Lemma 1. Hence the subring of $Q(KG)$ generated by $Q(KN)$ and KG is $\sum_{t \in T} Q(KN)t$, which has normalizing basis T over $Q(KN)$, and it is clearly a right quotient ring of KG . \square

Because of Lemma 2(b) we shall, whenever N is a normal subgroup of G , denote by $Q(KN) \cdot G$ the subring of $Q(KG)$ generated by $Q(KN)$ and KG . (This same ring is studied by Loudon [39] and Lawrence and Loudon [36] under the name $Q(KN) \otimes_{KN} KG$ - our notation is more in keeping with the identification of Lemma 1.)

We shall often find the ring $Q(KN) \cdot G$ useful as a sort of halfway house between KG and $Q(KG)$. As Lemma 2 shows, the basic relationships between N and G (namely, the closure of N under conjugation by elements of G , and the decomposition of G into cosets of N) have been translated fairly accurately into relationships between the rings $Q(KN)$ and $Q(KN) \cdot G$. On

the other hand, we have the advantages (especially in the nonsingular case) of the nicer structure of the MRQ ring $Q(KN)$. Our next result illustrates these points.

Proposition 3. Suppose N is a normal subgroup of G . If KG is nonsingular with uniform right ideals then so is KN .

Proof. For clarity's sake we introduce the temporary notation $R = Q(KN)$, $S = Q(KN) \cdot G$ and $Q = Q(KG)$. Since KG is nonsingular, Q is regular and so a uniform right ideal of Q is just a minimal right ideal of Q . But, because Q is a right quotient ring of KG , any uniform right ideal of KG generates a uniform right ideal of Q . Hence we can find a nonzero $s \in S$ such that sQ is a minimal right ideal of Q . Write $s = s_1 x_1 + \dots + s_n x_n$ where each $s_i \in R$ and x_1, \dots, x_n are elements of some transversal for N in G . We may suppose that $x_1 = 1$ and that n is minimal for all these properties. Hence $\underline{\ell}_R(s_1) = \underline{\ell}_R(s_2) = \dots = \underline{\ell}_R(s_n)$ (else we could reduce n). It is enough to show that $s_1 R$ is a minimal right ideal of R , so suppose it is not minimal.

By Lemma 1.12, KN is nonsingular and so R is regular. Hence, as $s_1 R$ is not minimal, there is some $a \in R$ such that $s_1 a \neq 0$ and $\underline{\ell}_R(s_1) \subset \underline{\ell}_R(s_1 a)$. Say $b \in R$ with $bs_1 a = 0$ but $bs_1 \neq 0$. Then

$$sa = s_1 a + s_2 a_2 x_2 + \dots + s_n a_n x_n$$

where each $a_i = x_i a x_i^{-1}$ is in R (by Lemma 2(a)). As $s_1 a \neq 0$ we have $sa \neq 0$ (by Lemma 2(b)) and so $saQ = sQ$ (as sQ is a minimal right ideal of Q). By the minimality of n , $\underline{\ell}_R(s_1 a) = \dots = \underline{\ell}_R(s_n a_n)$ and so, as $bs_1 a = 0$, $bsa = 0$. As $saQ = sQ$, $bs = 0$.

But $b \in R$ and $bs = bs_1 + bs_2x_2 + \dots + bs_nx_n$ and so, by Lemma 2(b), $bs_1 = 0$. This contradiction completes the proof. \square

Remark. The above proof seems to rely heavily on the regularity of $Q(KN)$ and $Q(KG)$, and I do not know whether Proposition 3 remains true when KG is not nonsingular.

When N in Proposition 3 is locally finite our uniform right ideals become minimal right ideals and we can use the results from Chapter 2 about socle in group algebras.

Proposition 4. Suppose G is a group containing no nontrivial, finite, characteristic subgroups. If KG is nonsingular and has uniform right ideals then G contains no nontrivial, locally finite, normal subgroups.

Proof. Let N be the unique, maximal, locally finite, normal subgroup of G . By Proposition 3, KN is nonsingular and has uniform right ideals. Since N is locally finite, KN is semisimple (by Theorem 1.16) and locally Artinian. By Lemma 2.7, every uniform right ideal of KN is minimal and so $\text{soc } KN \neq 0$. By hypothesis, N has no nontrivial, finite, characteristic subgroups and so, by Proposition 2.16, $N = 1$. \square

Since a prime group contains no nontrivial, finite, characteristic subgroups, our main result now follows immediately from Proposition 4.

Theorem 5. Suppose KG is prime, nonsingular and has uniform right ideals. Then G contains no nontrivial, locally finite, normal subgroups. \square

Remark. This result is not true without the non-singularity of KG . For instance, if G is any prime, locally finite, p -group (say, $Z_p \setminus Z_{p^\infty}$) and if K is a field with $\text{char } K = p$ then KG has a uniform right ideal (namely, KG itself - see Example 4.3 of Brown [1]) yet G is itself locally finite.

Corollary 6. Suppose N is a subnormal (or even an ascendant) subgroup of G . If KG is prime nonsingular with uniform right ideals then so is KN .

Proof. Consider first the case where N is normal in G . Then KN is nonsingular with uniform right ideals by Lemma 1.12 and Proposition 3. Since $\Delta^+(N)$ is a locally finite, normal subgroup of G , $\Delta^+(N) = 1$ (by Theorem 5) and so KN is prime. The case where N is subnormal in G now follows by induction. Using transfinite induction we see that the result is also true when N is ascendant in G since the normal closure in G of a locally finite group ascendant in G is again locally finite. □

In [23] Handelman and Lawrence show that if KG is a strongly prime ring then G has no nontrivial, locally finite, normal subgroups, and they conjecture that the converse is also true. Since a strongly prime ring with uniform right ideals necessarily has a simple Artinian MRQ ring (see Corollary 1 on page 218 of Handelman and Lawrence [23]), this would suggest that, under the hypotheses of Theorem 5, $Q(KG)$ is always simple Artinian (in other words, the MRQ

ring of a group algebra is never an infinite dimensional full linear ring). Handelman and Lawrence's conjecture is clearly true if G is locally nilpotent (since then the torsion elements of G form a locally finite, normal subgroup of G , and since the group algebra of a torsion-free nilpotent group is a domain) or, as Brown has noted in [2], if G is FC-hypercentral. We shall see in Chapter 4 that their conjecture is also true if G is soluble but I have been unable to verify it for arbitrary G . Hence in the following sections we try a different approach to seeing whether $Q(KG)$ is in fact simple Artinian under the hypotheses of Theorem 5.

2. RESIDUALLY FINITE FACTOR GROUPS.

Suppose KG is prime, nonsingular and has uniform right ideals, and let N be a normal subgroup of G . By Corollary 6, $Q(KN)$ is at least a full linear ring. If we believe, as Handelman and Lawrence's conjecture says we should, that $Q(KG)$ is always simple Artinian, then Corollary 6 suggests the following induction procedure: suppose we have already shown that $Q(KN)$ is simple Artinian; let us examine conditions on G/N which will force $Q(KG)$ to be simple Artinian as well. In this section we look at the case where G/N is residually finite.

We need first the following technical results.

Lemma 7. Suppose N is a normal subgroup of G .

- (a) If $[G:N] < \infty$ then $Q(KN) \cdot G = Q(KG)$ and if, in addition, KN is nonsingular, then $Q(KG)$ is right self-injective.
- (b) If A is a right $Q(KN)$ -submodule of $Q(KG)$ and $g \in G$ then $Ag = \{ag : a \in A\}$ is a right $Q(KN)$ -submodule of

$Q(KG)$ with its lattice of submodules isomorphic to that of A .

Proof.

- (a) That $Q(KN) \cdot G = Q(KG)$ follows as in (3.2) of Burgess [5] or Corollary 12 of Louden [39]. If KN is nonsingular then $Q(KN)$ is right self-injective and hence so is $Q(KN) \cdot G$ (see (2.8) of Burgess [5]).
- (b) Define a second $Q(KN)$ -action on A by setting $a * q = agqg^{-1}$ for all $a \in A$ and $q \in Q(KN)$. By Lemma 2(a), $(A, *)$ has the same submodules as (A, \cdot) . Since (Ag, \cdot) is isomorphic to $(A, *)$ the result now follows. \square

The proof of the next lemma has been adapted from Lemma 4 of Hartley and Richardson [27] (where the same techniques were used to discuss $\text{soc } KG$ rather than $\text{soc } Q(KG)$).

Lemma 8. Suppose N is a normal subgroup of G such that $[G : N] < \infty$, and suppose $Q(KG)$ is semiprime. Then

$$\text{soc } Q(KG) = \text{soc } Q(KN) \cdot Q(KG)$$

and
$$\text{soc } Q(KN) = Q(KN) \cap \text{soc } Q(KG).$$

Proof. Suppose first that $a \in \text{soc } Q(KN)$ so that, by Lemma 7, $aQ(KG)$ is a finite direct sum of irreducible $Q(KN)$ -modules. Hence $aQ(KG)$ satisfies the minimum condition for $Q(KN)$ -submodules and so for $Q(KG)$ -submodules. As $Q(KG)$ is semiprime, every minimal right ideal is a direct summand of $Q(KG)$. Hence $aQ(KG)$ is a finite direct sum of irreducible $Q(KG)$ -modules and so $a \in \text{soc } Q(KG)$. Thus $\text{soc } Q(KN) \subseteq \text{soc } Q(KG)$.

Now suppose that $a = a_1 x_1 + \dots + a_n x_n \in \text{soc } Q(KG)$ where each $a_i \in Q(KN)$ and x_1, \dots, x_n form a transversal for N in G . To show that each $a_i \in \text{soc } Q(KN)$ it is enough to suppose that L is a large right ideal of $Q(KN)$ and show that each $a_i \in L$ (since the socle is the intersection of the large right ideals of $Q(KN)$). Since each Lx_i is then a large $Q(KN)$ -submodule of $Q(KN)x_i$, it follows that $L \cdot Q(KG) = Lx_1 + \dots + Lx_n$ is a large right ideal of $Q(KG) = Q(KN)x_1 + \dots + Q(KN)x_n$ (both sums being direct sums of $Q(KN)$ -modules). Hence $a \in Lx_1 + \dots + Lx_n$ and equating the components of a in this direct sum with the components from the expression $a = a_1 x_1 + \dots + a_n x_n$ gives each $a_i \in L$. Hence $\text{soc } Q(KG) \subseteq \text{soc } Q(KN) \cdot Q(KG)$.

Finally $Q(KN) \cap \text{soc } Q(KG) \subseteq \text{soc } Q(KN)$ follows from the previous paragraph with $x_1 = 1$ and $a_2 = \dots = a_n = 0$. \square

Proposition 9. Suppose N is a normal subgroup of G such that G/N is residually finite. If KG is nonsingular and has uniform right ideals then

$$Q(KN) \cap \text{soc } Q(KG) \neq 0.$$

Proof. By hypothesis, $\text{soc } Q(KG)$ is a nonzero ideal of $Q(KG)$. By Lemma 2(b) we can find a nonzero $a \in Q(KN) \cdot G \cap \text{soc } Q(KG)$ and write $a = a_1 x_1 + \dots + a_n x_n$ where each $a_i \in Q(KN)$ and Nx_1, \dots, Nx_n are distinct cosets of N in G . As G/N is residually finite there is a normal subgroup H of G containing N such that $[G:H] < \infty$ and Hx_1, \dots, Hx_n are distinct. Since KG is nonsingular, $Q(KG)$ is certainly semiprime and so, by Lemma 8,

$a \in \text{soc } Q(KG) = (Q(KH) \cap \text{soc } Q(KG)) \cdot Q(KG)$. If we equate the components for a given by this with the components of the expression $a_1 x_1 + \dots + a_n x_n$ (possible because Hx_1, \dots, Hx_n are distinct) we find that each $a_i \in \text{soc } Q(KG)$. Since $a \neq 0$ and each $a_i \in Q(KN)$ we thus have $Q(KN) \cap \text{soc } Q(KG) \neq 0$. \square

Notice here that our conclusion is weaker than the corresponding result in Hartley and Richardson [27] (where $KN \cap \text{soc } KG$ actually generates $\text{soc } KG$). Basically this is because in the group algebra situation we have $(KN) \cdot G = KG$ whereas, in general, $Q(KN) \cdot G \neq Q(KG)$ (see §3).

We are now ready for the main result in this section. As an unexpected bonus, we do not need to assume primeness here.

Theorem 10. Suppose KG is nonsingular and has uniform right ideals. If N is a normal subgroup of G such that G/N is residually finite and $Q(KN)$ is simple Artinian then $Q(KG)$ is semisimple Artinian.

Proof. By Proposition 9, $Q(KN) \cap \text{soc } Q(KG)$ is a nonzero ideal of $Q(KN)$ and so equals $Q(KN)$. Hence $1 \in \text{soc } Q(KG)$ and so $Q(KG)$ is semisimple Artinian. \square

An immediate corollary to Theorem 10 is:

Corollary 11. Suppose G is a residually finite group. If KG is nonsingular and has uniform right ideals then $Q(KG)$ is semisimple Artinian. \square

Remark. It is somewhat surprising that Corollary 11 should be true even when KG is not prime. Certainly it is easy to find nonsingular rings which have a uniform right ideal but which also have nonzero right ideals containing no uniform right ideals. We cannot, however, remove any other hypotheses in Corollary 11: if G is a noncyclic free group then G is residually finite but $Q(KG)$ is simple and directly infinite; nor is Corollary 11 true for arbitrary groups G (see §3).

3. LOCALLY FINITE FACTOR GROUPS.

In this section we consider essentially the same situation as in §2 but with locally finite, instead of residually finite, factor groups. Calculations based on Lemma 8 do not seem to be any help here and we turn instead to a generalization of the dimension function in Chapter 2. That function may be retrieved from what follows by setting $N = 1$.

Suppose N is a normal subgroup of G such that $Q(KN)$ is semisimple Artinian and G/N is locally finite. If X is a finitely generated right $Q(KN)$ -module let $\dim X$ be the composition length (= uniform dimension) of X . Let $R = Q(KN)$ and $S = Q(KN) \cdot G$, and denote by $L(S)$ the lattice of finitely generated right ideals of S . We define $d : L(S) \rightarrow [0, 1]$ as follows: if $s_1, \dots, s_n \in S$ choose a subgroup H of G containing N such that $[H : N] < \infty$ and $s_1, \dots, s_n \in Q(KN) \cdot H$, and write

$$d\left(\sum_{i=1}^n s_i S\right) = \left(\dim\left(\sum_{i=1}^n s_i Q(KN) \cdot H\right)\right) / \dim Q(KN) \cdot H$$

Proposition 12. Suppose N is a normal subgroup of G such that $Q(KN)$ is semisimple Artinian and G/N is locally finite. Then d is a well-defined, rational-valued function such that, for any $A, B \in L(S)$,

- (i) $d(A) \geq 0$ with equality if and only if $A = 0$,
- (ii) $d(B) \leq 1$ with equality if and only if $B = S$,
- (iii) if $A \cap B = 0$ then $d(A + B) = d(A) + d(B)$,
- (iv) if $A \lesssim B$ then $d(A) \leq d(B)$.

Proof. We note first that the expression for $d(\sum s_i S)$ makes sense (since $\dim Q(KN) \cdot H$ is finite) and is a rational number. That $d(\sum s_i S)$ is independent of the choice of H follows as it did for the dimension function in Proposition 2.1 by using Lemma 2(b) and Lemma 7(b) above. Independence of the choice of s_1, \dots, s_n then follows easily. Properties (i) to (iv) are consequences of the corresponding results for $\dim X$. □

The next lemma shows another property which $Q(KN) \cdot G$ shares with the locally finite group algebras of Chapter 2.

Lemma 13. Suppose KG is nonsingular and N is a normal subgroup of G such that $Q(KN)$ is semisimple Artinian and G/N is locally finite. Then $Q(KN) \cdot G$ is semisimple and locally Artinian.

Proof. Obviously $Q(KN) \cdot G$ is locally Artinian. Since KN is nonsingular, $Q(KN) \cdot H$ is right self-injective when H

is a subgroup of G containing N such that $[H : N] < \infty$ (by Lemma 7(a)). Hence the Jacobson radical and the right singular ideal of $Q(KN) \cdot H$ coincide (see page 47 of Faith [9]). Thus the same is true of $Q(KN) \cdot G$ (since G/N is locally finite) and so, as $Q(KN) \cdot G$ is right nonsingular (since KG is), $Q(KN) \cdot G$ must be semisimple. \square

Theorem 14. Suppose KG is prime, nonsingular and has uniform right ideals. If N is a normal subgroup of G such that $Q(KN)$ is simple Artinian and G/N is locally finite then $Q(KG) = Q(KN) \cdot G$ and is simple Artinian.

Proof. It is enough to show that $S = Q(KN) \cdot G$ is simple Artinian. By Lemma 13 and Lemma 2.7, every nonzero right ideal of S contains a nonzero idempotent. Since S has uniform right ideals we deduce that $\text{soc } S \neq 0$. Let $A \neq 0$ be a minimal right ideal of S . For any nonzero right ideal B of S we have $A \lesssim B$ (since S is prime) and so, by Proposition 12, $d(A) \leq d(B)$. Hence $d(A)$ is the least nonzero value attained by d . By Proposition 12, S has at most $1/d(A)$ independent nonzero right ideals and so S has finite uniform dimension. Being a prime Goldie ring with nonzero socle, S must be simple Artinian. \square

Remark. Unlike Corollary 11, Theorem 14 is not true if we drop the primeness of KG . For example, let K be the field of rational numbers, $G = \mathbb{Z}_{p^\infty}$ the Prüfer p -group, and $N = 1$. Then KG is nonsingular (indeed, regular) and has uniform right ideals (in fact, by Satz 2.5 of Müller [40] we have $\text{soc } KG \neq 0$).

Also $Q(KN) = K$ is simple Artinian and G/N is locally finite. However $Q(KG)$ cannot be semisimple Artinian since G contains (in fact, is) an infinite locally finite subgroup (see (4.2) of Burgess [5]).

The equality $Q(KG) = Q(KN) \cdot G$ attained in Theorem 14 is unusual and we shall now depart momentarily from our main theme to discuss it further.

Suppose N is a normal subgroup of G . We saw in Lemma 7 that if $[G : N] < \infty$ then $Q(KG) = Q(KN) \cdot G$. One might be tempted to conjecture that the converse is also true (for instance, if $N = 1$ and KG is nonsingular then $Q(KG) = Q(KN) \cdot G$ only if KG is self-injective and so G is finite) but the following simple counter-example is given by Lawrence and Loudon in [36]: Let $G = \prod_I \mathbb{Z}$ (where I is some infinite index set) and let $N = \prod_I n \mathbb{Z}$ (where $n > 1$) so that KG is a commutative integral domain (since G is torsion-free abelian); clearly $[G : N] = \infty$ yet $Q(KG) = Q(KN) \cdot G$ (this follows, for example, from Theorem 14). So instead Lawrence and Loudon propose the following conjecture (which generalizes a suggestion of K.A. Brown; see [36]):

Conjecture. Suppose N is a normal subgroup of G . Then $Q(KG) = Q(KN) \cdot G$ if and only if either

- (i) $[G : N] < \infty$, or
- (ii) G/N is locally finite and both $Q(KN)$ and $Q(KG)$ are Artinian classical quotient rings.

To support this conjecture they prove it in the following special cases:

- (a) KG is semiprime, singular and countable (see Theorem A of [36]),
- (b) KG is semiprime and countable, and G/N is locally finite (see Proposition 13 of [36]),
- (c) $N(KG)$, the sum of the nilpotent ideals of KG , is dense in KG , and KG is countable (see Remark 4 in Section 6 of [36]).

Notice that in each case KG is countable.

We have already looked at one special case of this conjecture in §4 of Chapter 2: if we put $N = 1$, the conjecture says that $KG = Q(KG)$ (that is, KG is rationally complete) if and only if G is finite. We proved this, under the added hypothesis that G is locally finite, in Proposition 2.21. We shall now look at another case where the dimension function allows us to drop the countability assumed by Lawrence and Louden.

Proposition 15. Suppose KG is nonsingular and N is a normal subgroup of G with G/N locally finite. Then $Q(KG) = Q(KN) \cdot G$ if and only if either

- (i) $[G : N] < \infty$, or
- (ii) $Q(KG)$ is semisimple Artinian.

Proof. As before, write $S = Q(KN) \cdot G$ and note that, as KG is nonsingular, $S = Q(KG)$ if and only if S is right self-injective.

⇐. If $[G : N] < \infty$ then $Q(KG) = Q(KN) \cdot G$ by Lemma 7(a), so suppose $Q(KG)$ is semisimple Artinian. Thus KG has finite

right uniform dimension. As N is normal in G , KN is non-singular (by Lemma 1.12) and has finite right uniform dimension. Hence $Q(KN)$ is semisimple Artinian. By Lemma 13 and Lemma 2.7, every uniform right ideal of S is an idempotent generated, minimal right ideal of S . As S has finite right uniform dimension it follows that $\text{soc } S = S$ and so S is right self-injective.

\Rightarrow . Suppose $[G : N] = \infty$ but $S = Q(KG)$. By Proposition 3.9 of Loudon [38], $Q(KN)$ is semisimple Artinian, and we thus have the function d from Proposition 12. We must show that S is semisimple Artinian, so suppose it is not. Being regular, S thus has a countably infinite sequence e_1, e_2, \dots of nonzero, orthogonal idempotents. As S is right self-injective there is, for each $I \subseteq \mathbb{N}$, an idempotent $e \in S$ such that eS is the injective envelope of $\sum_{n \in I} e_n S$. Suppose $I \subset J$ are distinct subsets of \mathbb{N} and e, f (respectively) generate the corresponding injective envelopes. As there is an uncountable family of subsets of \mathbb{N} linearly ordered by inclusion, and as the range of d is countable, the desired contradiction will follow if we show that $d(eS) \neq d(fS)$.

Choose $m \in J \setminus I$. Then $e_m S \cap \sum_{n \in I} e_n S = 0$ and so, as $\sum_{n \in I} e_n S$ is essential in eS , $e_m S \cap eS = 0$. Also, $e_m S + eS$, being a submodule of $(\sum_{n \in J \setminus I} e_n S) + eS$, can be embedded in fS since $(\sum_{n \in J \setminus I} e_n S) + eS$ is an essential extension of $\sum_{n \in J} e_n S$. By Proposition 12 we have, putting all this together,

$$\begin{aligned} d(eS) &< d(e_m S) + d(eS) \\ &= d(e_m S + eS) \\ &\leq d(fS) \quad \text{as required.} \end{aligned}$$

Hence $S = Q(KG)$ is semisimple Artinian. □

Remark 1. Since a nonsingular group algebra KH is semiprime (see Lemma 3.1 of Brown [1]), $Q(KH)$ is a classical quotient ring of KH whenever $Q(KH)$ is semisimple Artinian (see page 80 of Faith [9]). Hence Proposition 15 does indeed verify the above conjecture for the case where KG is nonsingular and G/N is locally finite.

Remark 2. Proposition 15 gives yet another proof that if G is locally finite and KG is self-injective then G is finite in the special case where KG is nonsingular (since if KG is Artinian, G is finite).

It is not known in general whether the equality $Q(KG) = Q(KN) \cdot G$ forces G/N to be locally finite. However, Lawrence and Louden have shown (in Proposition 14 of [36]) that if $Q(KN)$ and $Q(KG)$ are classical quotient rings then G/N is at least torsion, and if in addition $\text{char } K = 0$ then G/N is locally finite. Thus we can prove:

Corollary 16. When $\text{char } K = 0$ the above conjecture is true.

Proof. Suppose $Q(KG) = Q(KN) \cdot G$ and $\text{char } K = 0$, but $[G : N] = \infty$. By Theorem 1.14, KG is nonsingular and so, by Proposition 3.9 of Louden [38], $Q(KN)$ is semisimple Artinian. As in Remark 1 above, $Q(KN)$ is a classical quotient ring of KN and so, as $Q(KG) = Q(KN) \cdot G$, $Q(KG)$ is a classical quotient ring of KG . Since $\text{char } K = 0$, Proposition 14 of Lawrence and Louden [36] shows that G/N is locally finite, and the corollary now follows from Proposition 15. □

4. IS $Q(KG)$ EVER AN INFINITE DIMENSIONAL FULL LINEAR RING?

In this section we use the results of the previous sections to construct various classes of groups G for which $Q(KG)$ is simple Artinian whenever KG is prime, nonsingular and has uniform right ideals. We adopt the 'induction procedure' which motivated §§2, 3 and this leads us to consider the following class of groups.

Definition. Let C be the class of all groups X which satisfy the following condition:

If KG is prime, nonsingular and has uniform right ideals and if N is a normal subgroup of G such that $Q(KN)$ is simple Artinian and $G/N \cong X$ then $Q(KG)$ is simple Artinian.

Notice that if $X \in C$ then putting $N = 1$ in the above condition shows that $Q(KX)$ is simple Artinian whenever KX is prime, nonsingular and has uniform right ideals. However, this latter condition (although the true object of our interest) does not lend itself so easily to the constructive approach we use below.

Notice too that if X is any CC-group in C such that KX is prime nonsingular then $Q(KX)$ is always a simple ring: simple Artinian if KX has uniform right ideals, simple and directly infinite otherwise (by Theorem 1.13).

Handelman and Lawrence's conjecture about strongly prime group algebras (see §1) says that C should be the class of all groups. Although we do not succeed in proving this, we do find several large classes of groups in C . By Theorems 10 and 14, C contains all residually finite groups and all locally finite groups. Another class is given in

the next lemma, but firstly we recall some notation from Brown [3]:

Definitions. Let \mathcal{A}_0 denote the class of all torsion-free abelian groups. Then $B\mathcal{A}_0$ is the smallest class containing \mathcal{A}_0 which is closed under the operators L and \hat{P} . Also \mathcal{U} denotes the class of all groups having an ascending series whose factors are all either in $B\mathcal{A}_0$ or finite, and only finitely many of these factors are finite.

Lemma 17. C contains the class $B\mathcal{A}_0$.

Proof. It is enough to show that if N is a normal subgroup of a group G such that $Q(KN)$ is simple Artinian and $G/N \in B\mathcal{A}_0$ then
right uniform dimension of KN = right uniform dimension of KG ,
and this is done in the proof of Theorem A of Brown [3]. \square

In particular, C contains all torsion-free abelian groups and, more generally, all torsion-free locally nilpotent groups (since a torsion-free nilpotent group has a finite series with torsion-free abelian factors, and since $B\mathcal{A}_0$ is \hat{P} -closed).

More complicated examples of groups in C can be constructed using the following result.

Lemma 18. If X is a group with an ascending series whose factors lie in C then $X \in C$ (that is, C is \hat{P} -closed).

Proof. By Corollary 6 the proof (by transfinite induction on the length of the series for X) is trivial except possibly at limit ordinals. Suppose KG is prime, nonsingular and has uniform right ideals, and let N be a normal subgroup of G such that there is an ascending series

$$N = N_0 \trianglelefteq N_1 \trianglelefteq \dots \trianglelefteq N_\lambda = G$$

where λ is a limit ordinal. By induction and Corollary 6 we may suppose that, for each $\mu < \lambda$, $Q(KN_\mu)$ is simple Artinian. Since $KG = \bigcup_{\mu < \lambda} KN_\mu$ and since large right ideals 'go up' from each KN_μ to KG (as N_μ is ascendant in G), Lemma 1 shows that $\bigcup_{\mu < \lambda} Q(KN_\mu)$ is a right quotient ring of KG . Being a union of simple rings with the same identity, $\bigcup_{\mu < \lambda} Q(KN_\mu)$ is simple and hence so is $Q(KG)$ as required. \square

We can now construct several familiar classes of groups which lie in C :

Theorem 19. C contains the following classes of groups:

- (a) soluble groups,
- (b) locally nilpotent groups,
- (c) FC-groups,
- (d) \mathcal{U} (defined above), hyperabelian, radical, FC-soluble or FC-hypercentral groups.

Proof. Since C contains all locally finite groups and all torsion-free abelian groups (by Theorem 14 and Lemma 17), it contains all soluble groups and all FC-groups (by Lemma 18). Since the torsion elements of a locally nilpotent group form a locally finite normal subgroup, and since the

corresponding factor group is a torsion-free locally nilpotent group, C contains all locally nilpotent groups (by Theorem 14 and Lemmas 17 and 18). Finally the groups in (d) all have an ascending series whose factors lie in C and so Lemma 18 completes the proof. \square

Before we consider our final class of groups we need the following lemma.

Lemma 20. Suppose KG is nonsingular and G contains a noncyclic free subgroup. Then $Q(KG)$ is directly infinite.

Proof. If H is a noncyclic free subgroup of G then KH is a non-Ore domain and so there is a non-zero-divisor $a \in KH$ such that aKH is not large in KH . Hence $\underline{r}_{KG}(a) = 0$ but aKG is not a large right ideal of KG . Thus a has a left inverse, but no right inverse, in $Q(KG)$ and $Q(KG)$ is directly infinite. \square

Proposition 21. Suppose KG is prime, nonsingular and has uniform right ideals, and let N be a normal subgroup of G such that $Q(KN)$ is simple Artinian and G/N is a linear group. If $\text{char } K = 0$ then $Q(KG)$ is simple Artinian.

Proof. If G/N is soluble-by-locally-finite then the result holds regardless of $\text{char } K$ (by Theorems 14 and 19 and Lemma 18). Otherwise G/N must contain a noncyclic free group (by Theorem 10.17 of Wehrfritz [53]). Hence G contains a noncyclic free group. Say $x, y \in G$ with $\langle x, y \rangle$ free but

not cyclic. Choose some nonzero $a \in KG$ with aKG a uniform right ideal and let H be the subgroup of G generated by N, x, y and $\text{supp } a$. Then aKH is a uniform right ideal of KH and, since $\text{char } K = 0$, KH is nonsingular by Theorem 1.14. But $Q(KN)$ is simple Artinian and H/N , being a finitely generated linear group, is residually finite (by Theorem 4.2 of Wehrfritz [53]). Hence, by Theorem 10, $Q(KH)$ is semisimple Artinian. Since H contains a noncyclic free subgroup this contradicts Lemma 20. This completes the proof. □

I do not know whether Proposition 21 remains true when $\text{char } K > 0$. It would certainly be true (as the above proof shows) if KG could not be prime, nonsingular with uniform right ideals when G contains a noncyclic free subgroup. On the other hand, if there is a group G containing a noncyclic free group such that KG is prime, nonsingular and has uniform right ideals then $Q(KG)$ is an infinite dimensional, full linear ring by Lemma 20.

Our results in this chapter show that for several large classes of groups the MRQ ring of a group algebra cannot be an infinite dimensional full linear ring. It is perhaps too soon to conjecture that this is always the case but I have been unable to construct a counterexample. (However, it is not difficult to construct an example of a torsion-free semigroup S whose semigroup algebra $K[S]$, for some field K , has an infinite dimensional full linear MRQ ring. For instance, let V be an infinite dimensional

vector space over a field K and let S be the multiplicative sub-semigroup of $\text{End}_K V$ generated by the elements represented by the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then KS is such an example.)

It is perhaps worth noting that if such a counterexample exists then the division ring involved must have infinite dimension over its centre:

Proposition 22. Suppose $Q(KG)$ is a full linear ring $\text{End}_D V$ (where V is a vector space over the division ring D). If D has finite dimension over its centre then $Q(KG)$ is simple Artinian.

Proof. We shall use Passman's terminology for generalized polynomial identities in [42]. Let e be a primitive idempotent of $Q = Q(KG)$ so that $eQe \cong D$ satisfies a standard polynomial identity s_n of degree n (for some integer n). Choose $b \in KG$ such that $0 \neq eb \in KG$ and let $a = eb$. Let f be the polynomial

$$f(x_1, \dots, x_n) = \sum_{\sigma \in S_n} (\text{sign } \sigma) ax_{\sigma(1)} ax_{\sigma(2)} a \dots ax_{\sigma(n)} a$$

where S_n is the symmetric group on n letters. Then f is multilinear. Since KG is semiprime and $a \neq 0$, we have

$(aKG)^{n+1} \neq 0$ and so, for each $\sigma \in S_n$,

$f_\sigma(x_1, \dots, x_n) = ax_{\sigma(1)}a \dots ax_{\sigma(n)}a$ is not an identity for KG . Hence f is non-degenerate. Finally if

$x_1, \dots, x_n \in KG \subseteq Q$ then

$$\begin{aligned} f(x_1, \dots, x_n) &= \sum_{\sigma \in S_n} (\text{sign } \sigma) ax_{\sigma(1)}a \dots ax_{\sigma(n)}a \\ &= \sum_{\sigma \in S_n} (\text{sign } \sigma) e(bx_{\sigma(1)})e \dots e(bx_{\sigma(n)})eb \\ &= s_n(ebx_1e, \dots, ebx_ne) \cdot b \\ &= 0. \end{aligned}$$

Hence KG satisfies the non-degenerate, multilinear, generalized polynomial f and so, by Theorem 3.3 of Passman [42], $[G : \Delta(G)] < \infty$. Since G is prime, $\Delta(G)$ is torsion-free abelian and so G is abelian-by-finite. By Theorem 19 (for instance), Q is thus simple Artinian. \square

Of course, examples where $Q(KG)$ is simple Artinian are easy to find. For instance, if $G = \mathbb{Z} \wr \mathbb{Z}_n$ (where n is any positive integer) then G is abelian-by-finite (indeed, it is also soluble, residually finite and linear) and $Q(KG)$ is simple Artinian. It seems unlikely that wreath products could also give examples where $Q(KG)$ is an infinite dimensional full linear ring since we have:

Proposition 23. Suppose G is the standard wreath product $A \wr B$ where A and B are nontrivial groups and B is infinite. If KG is nonsingular and has a uniform right ideal then KA is an Ore domain.

Proof. Let N be the base group of G so that N is normal in G and $N \cong \prod_B A$. By Proposition 3 KN is nonsingular and has a uniform right ideal, say aKN (where $0 \neq a \in KN$). There is a finite subset $B' \subseteq B$ such that $\text{supp } a \subseteq \prod_{B'} A$ (via the above isomorphism). Let $M = \prod_{B \setminus B'} A$ so that, as B is infinite, $M \cong N$. We show that KM is an Ore domain. Firstly suppose b, c are nonzero elements of KM and $bc = 0$. Then $ac = ca \neq 0$ (since $\langle \text{supp } a \rangle \cap M = 1$ and M centralizes $\langle \text{supp } a \rangle$), and so if $Q = Q(KN)$, $caQ = acQ = aQ$ since aQ is a minimal right ideal of Q . As $bc = 0$ we thus have $ba = 0$ which is impossible. Hence KM is a domain. By Proposition 3, KM has a uniform right ideal (since M is subnormal in G). Thus KM is an Ore domain. \square

In particular, if A were abelian in Proposition 23 we could use Passman's intersection theorem (see Lemma 21.1 of Passman [41]) to deduce that $Q(KG)$ is simple Artinian.

A related problem to the existence of infinite dimensional full linear MRQ rings of group algebras is that of determining when group algebras have nonzero socle (indeed Richardson's discussion of this problem in [48] inspired the definition of C above). If a prime group algebra has nonzero socle its MRQ ring must be an infinite dimensional full linear ring (if the group algebra were a prime Goldie ring then, having nonzero socle, it would be simple Artinian and so trivial). However Richardson conjectured that the group algebra of a non-locally-finite group always has zero socle. In particular it would follow from Theorem 2.4 that prime group algebras always have zero socle.

As for the actual existence of uniform right ideals in group algebras, very little is known. All previous work (that I know of) has concentrated on criteria for the existence of Artinian classical quotient rings for group algebras and even this problem seems very difficult. We have already mentioned the best known necessary condition that $Q(KG)$ be Artinian: G has no infinite locally finite subgroups (see (4.2) of Burgess [5]). The weakest known sufficient condition is that $G \in \mathcal{U}$ (defined earlier) but this does not even account for all soluble groups G for which $Q(KG)$ is Artinian (see Theorem A of Brown [3] and Example 4.5 of Brown [4]). Finally Lewin [37] has shown that if G is soluble such that KG is a domain then KG is an Ore domain (and so KG is itself uniform) but this, of course, brings us into contact with yet another unsolved conjecture: the zero-divisor conjecture.

CHAPTER 4

SOLUBLE GROUP ALGEBRAS

SUMMARY

We now return to the study of general MRQ rings of prime nonsingular group algebras. In §1 we discuss the presence of countable insulators in prime group algebras and use them to generalize Theorem 1.13:

Theorem 3. Suppose KG is prime nonsingular and N is a normal subgroup of G . If N is a CC-group and KN has no uniform right ideals then $Q(KG)$ is directly infinite and, for any proper ideal I of $Q(KG)$, $I \cap KN = 0$.

When N can be chosen so that a suitable intersection theorem holds we can say even more, and we develop such a theorem for certain types of wreath products.

In §2 we turn to group algebras of soluble groups where, of course, we have Zalesskii's well-known intersection theorem. We use this to verify Handelman and Lawrence's conjecture about strongly prime group algebras when the group concerned is soluble. Applying Theorem 3 we obtain:

Theorem 10. Suppose G is soluble and KG is prime nonsingular. Then either KG is strongly prime or $Q(KG)$ is simple and directly infinite.

In particular it follows that $Q(KG)$ in Theorem 10 is always simple and right self-injective. We conclude by

discussing which of the three kinds of simple, right self-injective ring studied by Goodearl and Handelman (namely, simple Artinian, or simple and directly infinite, or simple and directly finite but not Artinian) can occur as $Q(KG)$ when G is soluble.

1. COUNTABILITY IN GROUP ALGEBRAS: A REFINEMENT.

In this section we find a countability condition satisfied by all prime group algebras and use it to improve Theorem 1.13. This condition is given by our first result:

Lemma 1. Suppose KG is a prime group algebra. Each nonzero $a \in KG$ has a countable left (or right) insulator in G (that is, there is a countable subset X of G such that $\ell_{-KG}(xa : x \in X) = 0$).

Proof. Since $\text{supp } a$ is finite and $\Delta^+(G) = 1$, it is easy to construct a countable subgroup H of G such that $\text{supp } a \subseteq H$ and $\Delta^+(H) = 1$. Then KH is a prime ring and so $\ell_{-KH}(xa : x \in H) = 0$. Because KG is free as a right KH -module it follows that $\ell_{-KG}(xa : x \in H) = 0$ and so H will do as the left insulator. □

Suppose now that KG is a prime, nonsingular group algebra. Since $Q(KG)$ is a right quotient ring of KG , it follows from Lemma 1 that each nonzero element of $Q(KG)$ has a countable right insulator. As the existence of finite right insulators would force $Q(KG)$ to be simple ($Q(KG)$ would be strongly prime; see Handelman and Lawrence [23],

Proposition II.3), it would be reasonable to expect that having countable insulators would restrict $Q(KG)$ to having at most one proper ideal. However, there are prime, regular, right self-injective rings Q in which each nonzero element has a countable right insulator but which have arbitrarily many ideals. For instance, if Q is the full linear ring $\text{End}_K V$ (where V is a vector space over the field K , and endomorphisms are written on the left) then each nonzero element of Q has a countable right insulator if and only if $\dim_K V \leq |K|^{\aleph_0}$ so that, by choosing K large enough, we can have as many ideals in Q as we like.

It is interesting to note that such a full linear ring has countable left insulators for each of its nonzero elements if and only if $\dim_K V \leq \aleph_0$. In particular such a ring would have at most one proper ideal. In fact, it is not too hard to show, using Goodearl's description of the ideals of a prime, regular, right self-injective ring Q (see Theorem 1.1), that if each nonzero element of Q has a countable left insulator then $H(\aleph_0)$ is the only possible proper ideal of Q . However, there is no guarantee that $Q(KG)$ should inherit left insulators from KG : for instance, if G contains a non-cyclic free subgroup, then KG contains elements a with $\ell_{KG}(a) = 0 = \underline{r}_{KG}(a)$ but $\ell_{Q(KG)}(a) \neq 0$.

Nevertheless, it is the left insulators which we shall use. Firstly we need an easy lemma.

Lemma 2. Let R be any ring, and let A and B be right R -modules. If B is nonsingular and $\text{Hom}_R(A, B) \neq 0$ there are nonzero submodules $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cong B'$.

Proof. Let $f : A \rightarrow B$ be a nonzero R -homomorphism. As $f(A)$ is a nonsingular R -module, $\ker f$ cannot be a large submodule of A . Hence there is a nonzero submodule A' of A such that $A' \cap \ker f = 0$. Putting $B' = f(A') \cong A'$ then completes the proof. \square

We can now prove the main result of this section, a generalization of Theorem 1.13.

Theorem 3. Suppose KG is prime nonsingular and N is a normal subgroup of G . If N is a CC-group and KN has no uniform right ideals then $Q(KG)$ is directly infinite and, for any proper ideal I of $Q(KG)$, $I \cap KN = 0$.

Proof. Let $Q = Q(KG)$. It is enough to show that for any nonzero $a \in KN$ there is a large right ideal L of KG such that $L \lesssim aKG$. Then choosing a so that aKN is not a large right ideal of KN shows that Q is directly infinite. Furthermore for any nonzero $a \in KN$ we would have $Q \lesssim aQ$ and so $QaQ = Q$ which gives the rest of the result.

So we suppose that $0 \neq a \in KN$. Because KN has no uniform right ideals there are nonzero elements a_1, a_2, \dots in aKN such that the right ideals a_1KN, a_2KN, \dots are independent. As KG is prime we can, by Lemma 1, find a countable subset X of G such that, for each positive integer n , $\ell_{KG}(xa_n : x \in X) = 0$. In particular we have, for each n , $\ell_{KN}(a_n^x : x \in X) = 0$ where, of course, $a_n^x = xa_nx^{-1}$.

Now let

$$H = \langle g^x : x \in X, g \in \bigcup_{n=1}^{\infty} \text{supp } a_n \rangle^N$$

which is a countable normal subgroup of N because N is a CC-group. Let F be the subfield of K generated by the coefficients of all the a_n . Then FH is a countable nonsingular ring (nonsingular by Lemmas 1.7 and 1.12 since F is a subfield of K and H is subnormal in G). Suppose b_1, b_2, \dots is an enumeration of the nonzero elements of FH .

We now use an inductive construction similar to that in Theorem 1.5 to find a large right ideal L of FH such that $L.KG \lesssim a.KG$. (It should be noted however that, in general, we will not have $L \lesssim a.FH$.)

Step 1. As $b_1 \neq 0$ there is some $x \in X$ such that $b_1 a_1^x \neq 0$. Hence there is a nonzero FH -homomorphism $a_1^x FH \rightarrow b_1 FH$ (given by left multiplication by b_1). As $b_1 FH$ is nonsingular there is, by Lemma 2, some nonzero $b'_1 \in b_1 FH$ such that $b'_1 FH \lesssim a_1^x FH$. Because K is free over F and H is a subgroup of G , we get $b'_1 KG \lesssim a_1^x KG \cong a_1 KG$, this last isomorphism being the one which need not hold in FH .

Induction Step. Suppose we have found b'_1, \dots, b'_n in FH satisfying:

- (a) $b'_1 FH, \dots, b'_n FH$ are independent;
- (b) for each i ($1 \leq i \leq n$), $b'_i KG \lesssim a_i KG$;
- (c) for each i ($1 \leq i \leq n$), $b'_i FH \cap \sum_{j=1}^n b'_j FH \neq 0$.

If $b'_{n+1} FH \cap \sum_{j=1}^n b'_j FH \neq 0$ then we put $b'_{n+1} = 0$ to complete this step. Otherwise we choose some $x \in X$ such that $b'_{n+1} a_{n+1}^x \neq 0$ and proceed as in Step 1 to find some nonzero

$b'_{n+1} \in b_{n+1}FH$ such that $b'_{n+1}KG \lesssim a_{n+1}KG$. This completes the induction.

By construction $\sum_{n=1}^{\infty} b'_n FH$ is a large right ideal of FH . As F is a subfield of K and H is subnormal in G , $\sum_{n=1}^{\infty} b'_n KG$ must therefore be a large right ideal of KG . But, also by construction ((a) and (b) above), we have

$$\sum_{n=1}^{\infty} b'_n KG \lesssim \sum_{n=1}^{\infty} a_n KG \subseteq aKG$$

and so the proof is complete. □

By putting $N = G$, we can in fact now retrieve Theorem 1.13.

Remark. We could, instead of assuming that N is a CC-group in Theorem 3, adopt the hypothesis: for any countable subset X of N there is a countable subgroup H of N containing X and ascendant in N (all that is really needed is that H be countable and large right ideals of KH generate large right ideals of KN). In particular, Theorem 3 would still be true if N were, instead of a CC-group, a group satisfying the normalizer condition (that is, if every proper subgroup of N were properly contained in its normalizer in N). A corollary to this would be:

Proposition 4. Suppose G satisfies the normalizer condition. If KG is prime nonsingular then $Q(KG)$ is $SP(1)$, that is either a division ring or a simple, directly infinite ring.

Proof. If KG has no uniform right ideals then $Q(KG)$ is simple and directly infinite by the analogue of Theorem 3. Otherwise G has no nontrivial, locally finite, normal subgroups (by Theorem 3.5) and so, as G is locally nilpotent, G is a torsion-free, locally nilpotent group. Thus KG is an Ore domain. \square

Theorem 3 is most useful when the subgroup N can be chosen so that a suitable intersection theorem holds. One case where this is possible can be found using the following results (which are proved by imitating Lemma 7 and Proposition 2 of Zaleskii [54]).

Lemma 5. Suppose H is a subgroup of G and H_0 is a subgroup of countable index in H . Suppose $a \in KG$ and, for any $x \in H_0$, $ax = xa$. Then $\text{supp } a \subseteq \{g \in G : [H : C_H(g)] \leq \aleph_0\}$.

Proof. Write $a = a_1 g_1 + \dots + a_n g_n$ where $0 \neq a_i \in K$ and $g_i \in G$ for each i . For any $x \in H_0$, $\sum_{i=1}^n a_i x g_i = \sum_{i=1}^n a_i g_i x$ and so for each i there is some $j = j(i, x)$ such that $g_i x = x g_j$. Let s_{ij} be a representative of the class $C_{H_0}(g_i) s_{ij}$ of those elements $x \in H_0$ for which $g_i x = x g_j$ (when this class is non-empty). For each i we have $H_0 = \bigcup_j C_{H_0}(g_i) s_{ij}$ and so $C_{H_0}(g_i)$ has finite index in H_0 . Hence each $C_{H_0}(g_i)$ has countable index in H and the result follows. \square

Lemma 6. Suppose N is a normal CC-subgroup of G . If I is a nonzero ideal of KG there is some nonzero $a \in I$ with $\text{supp } a \subseteq \{g \in G : [N : C_N(g)] \leq \aleph_0\}$.

Proof. We choose a nonzero element $a = a_1 + a_2 g_2 + \dots + a_n g_n \in I$ where each $a_i \in KN$ and $g_i \in G$, and n is minimal for all this. Let $H_0 = \cap \{C_N(g) : g \in \text{supp } a\}$ which has countable index in N since $\text{supp } a$ is finite and N is a CC-group. For any $x \in H_0$ we have $xax^{-1} = a_1 + (xa_2 g_2 x^{-1} g_2^{-1}) g_2 + \dots + (xa_n g_n x^{-1} g_n^{-1}) g_n$. By the minimality of n we have $xax^{-1} - a = 0$ and so $xa = ax$. The result now follows from Lemma 5. \square

Proposition 7. Suppose N is a normal subgroup of G such that $N = \{g \in G : [N : C_N(g)] \leq \aleph_0\}$. If I is a nonzero ideal of KG then $I \cap KN \neq 0$. Hence if KG is prime nonsingular and KN has no uniform right ideals, $Q(KG)$ is simple and directly infinite.

Proof. The first assertion follows from Lemma 6. As N is, by hypothesis, a CC-group the second assertion is an immediate corollary to Theorem 3. \square

Example. We can use Proposition 7 to find $Q(KG)$ when G is a certain type of wreath product. Suppose A is a nontrivial CC-group and B is any infinite group. Suppose at least one of A and B is uncountable and let $G = A \wr B$ so that, by Lemma 21.5(iv) of Passman [41], G is a prime group. Let N be the base group of G so that $N \cong \prod_B A$ and G is a split extension of N by B . Then N is a CC-group (since A is) and so $N \subseteq \{g \in G : [N : C_N(g)] \leq \aleph_0\}$. On the other hand, since either A or B is uncountable, each nontrivial element of B has uncountably many N -conjugates and so $N = \{g \in G : [N : C_N(g)] \leq \aleph_0\}$. Thus the intersection theorem

from Proposition 7 is available. In particular KG is nonsingular if and only if KN is.

Let K be a field such that KG is nonsingular. If KN has no uniform right ideals (for instance, if A is locally finite) then, as G is prime, Proposition 7 shows that $Q(KG)$ is simple and directly infinite. Otherwise, the proof of Proposition 3.23 shows that KN is an Ore domain (for example, A could be torsion-free abelian). In this case, Proposition 7 shows that each nonzero ideal of KG contains a non-zero-divisor and so KG is strongly prime. Thus, in either case, $Q(KG)$ is simple.

Of course, if A and B are both countable so is G and Proposition 7 is no longer available. However in this case $Q(KG)$ must be either full linear or simple and directly infinite (by Theorem 1.13).

2. SOLUBLE GROUP ALGEBRAS.

Our main aim in this section is to show that, when KG is prime nonsingular and G is soluble, $Q(KG)$ is a simple ring. A partial result in this direction was proved in Chapter 3: if G is soluble then $Q(KG)$ cannot be an infinite dimensional full linear ring (by Theorem 3.19). Our methods here will in fact provide an alternative proof of this result.

We begin by determining which group algebras of soluble groups are strongly prime. We discussed in Chapter 3 Handelman and Lawrence's conjecture that the group algebra KG is strongly prime if and only if G has no nontrivial, locally finite, normal subgroups. We now verify their conjecture in the case where G is soluble (this result has

also been obtained by Brown [2]). We denote by $3(G)$ the Zaleskii subgroup of the soluble group G , so that $3(G)$ is an FC-group normal in G , and for each nonzero ideal I of KG , $I \cap K[3(G)] \neq 0$ (see Passman [43], p.81 and Zaleskii [54]).

Proposition 8. Let G be a soluble group. The following conditions are equivalent:

- (a) KG is strongly prime;
- (b) G has no nontrivial, locally finite, normal subgroups;
- (c) $3(G)$ is torsion-free abelian.

Proof.

(a) \Rightarrow (b): is proved by Handelman and Lawrence [23], Proposition III.1(a).

(b) \Rightarrow (c): follows because $3(G)$ is an FC-group whose torsion subgroup is normal in G .

(c) \Rightarrow (a): If I is a nonzero ideal of KG then, by the Theorem of Zaleskii [54], I contains a nonzero element of $K[3(G)]$. By hypothesis $K[3(G)]$ is an integral domain and so I contains a non-zero-divisor of KG . Hence KG is strongly prime. □

We are going to use Theorem 3 to handle general prime nonsingular group algebras of soluble groups, and for this we need to know when there are uniform right ideals in normal subgroup algebras. Our next lemma gives a useful necessary condition.

Lemma 9. Suppose KG is prime nonsingular and N is a normal subgroup of G . If KN has uniform right ideals then N contains no nontrivial, locally finite, normal subgroups.

Proof. As G is prime, N has no nontrivial, finite, characteristic subgroups and so the result is an immediate consequence of Proposition 3.4. □

Theorem 10. Suppose G is soluble and KG is prime and nonsingular. Then either KG is strongly prime or $Q(KG)$ is simple and directly infinite.

Proof. If $3(G)$ is torsion-free abelian then KG is strongly prime by Proposition 8 (indeed, the proof of (c) \Rightarrow (a) shows that KG is a two-sided order in a simple ring since we can form the ring of fractions with denominators in $K[3(G)] \setminus 0$). Otherwise $3(G)$ has a nontrivial, locally finite normal subgroup (namely, its torsion subgroup, since $3(G)$ is an FC-group) and so $K[3(G)]$ has no uniform right ideals (by Lemma 9). By Theorem 3 (putting $N = 3(G)$), $Q(KG)$ is directly infinite and no proper ideal of $Q(KG)$ contains a nonzero element of $K[3(G)]$. By the intersection Theorem of Zalesskii [54], $Q(KG)$ is simple. □

Since a strongly prime ring has a simple right self-injective MRQ ring, we can deduce immediately:

Corollary 11. If KG is prime nonsingular and G is soluble then $Q(KG)$ is simple and right self-injective. □

Remark 1. It should be noted that each possibility in Theorem 10 can occur. If G is any torsion-free abelian group then KG is strongly prime but $Q(KG)$ is not directly infinite. On the other hand, any locally finite, soluble group G for which KG is prime nonsingular (for example, $G = A \setminus B$ where A is a nontrivial, torsion, abelian group, with no elements of order $\text{char } K$, and B is an infinite, torsion abelian group) gives a simple, directly infinite $Q(KG)$ for which KG is not strongly prime (by Proposition 8(b)). In particular, we see that a ring need not be strongly prime when its MRQ ring is simple (see Example (b), p.826 of Goodearl and Handelman for another example of this kind). Notice too that the possibilities in Theorem 10 are not mutually exclusive: if A is a nontrivial, torsion-free, abelian group and if B is an infinite, torsion, abelian group (A and B both countable) then the soluble group $G = A \setminus B$ gives a strongly prime group algebra KG whose MRQ ring is simple and directly infinite.

Remark 2. We drew attention in Chapter 2 to the parallel between our result (Theorem 2.9) for MRQ rings of prime nonsingular group algebras of locally finite CC-groups and Fisher and Snider's result (in [11]) which shows that prime nonsingular group algebras of countable, locally finite groups are primitive. Unlike our result, which extends to arbitrary soluble locally finite groups (by Theorem 10), Fisher and Snider's breaks down even for uncountable metabelian groups (as Domanov's example shows; see Domanov [8] and Passman [46]).

Suppose G is a soluble group such that KG is prime nonsingular, and let $Q = Q(KG)$. By Corollary 11, Q is simple and right self-injective and so there are three possibilities: Q must be either (i) simple Artinian, or (ii) simple and directly infinite, or (iii) simple, directly finite but not Artinian (that is, $SP(\infty)$). We gave examples of (i) and (ii) in Remark 1. I do not know whether Q can be $SP(\infty)$, but I consider below the form a "minimal" such example must take. Firstly we have another corollary to Theorem 3.

Proposition 12. Suppose KG is prime nonsingular and $Q(KG)$ is directly finite. If N is a subnormal subgroup then KN is prime nonsingular and $Q(KN)$ is directly finite.

Proof. Suppose first that N is normal in G . Then we already know that KN is nonsingular (Lemma 1.12) and $Q(KN)$ is directly finite (since $Q(KN) \subseteq Q(KG)$ by Lemma 3.1). Now $\Delta^+(N)$ is a normal subgroup of G and is certainly a CC-group. As $Q(KG)$ is directly finite, Theorem 3 says that $K[\Delta^+(N)]$ has uniform right ideals and so, by Lemma 9, $\Delta^+(N) = 1$ since $\Delta^+(N)$ is itself locally finite. Thus KN is also prime. The result now follows by induction. \square

Now suppose that G is a soluble group such that $Q(KG)$ is $SP(\infty)$, and suppose that G has minimal solubility class for this condition. By Proposition 12 (or by Theorem 10), $Q(KG')$ is simple, right self-injective and directly finite. By the minimality of the class of G , $Q(KG')$ is not $SP(\infty)$

and so must be Artinian. Suppose $G' \subseteq H \subseteq G$ where H/G' is the torsion subgroup of the abelian group G/G' . If $Q(KH)$ were simple Artinian then, as G/H is torsion-free abelian, $Q(KG)$ would also be simple Artinian (by the proof of Lemma 3.17). Hence, by Proposition 12, $Q(KH)$ is $SP(\infty)$. We may thus assume that $N \trianglelefteq G$ such that $Q(KN)$ is simple Artinian and G/N is torsion abelian.

As G/N is abelian we can use Proposition 12 to assume G/N is countable (we just have to choose a countable extension H of N such that KH has an infinite direct sum of nonzero right ideals). Writing G/N as an ascending union of finite subgroups we see (by Proposition 10 again) that the right quotient ring $Q(KN).G$ of KG is an ascending union of simple Artinian rings with the same identity. Hence we ask the following question:

(*)... Suppose S is a ring with identity 1 containing subrings $S_1 \subseteq S_2 \subseteq \dots$ such that each S_n is simple Artinian containing 1, and $\bigcup_n S_n = S$.
Is $Q(S)$ ever $SP(\infty)$?

Such a ring S has a unique rank function N (since each S_n is simple Artinian). Thus, by Corollary 5.6 of Goodearl and Handelman [18], $Q(S)$ is $SP(\infty)$ if and only if S has zero socle and, for each large right ideal L of S , $\sup\{N(x) : x \in L\} = 1$. If the embeddings $S_n \rightarrow S_{n+1}$ are such that each S_n is centralized by some nontrivial idempotent of S_{n+1} , then the construction in Goodearl and Handelman's Example (e) (page 831 of [18]) gives a large right ideal L of S with $\sup\{N(x) : x \in L\} < 1$ and so $Q(S)$ is simple and

directly infinite. However, in our case (where $S = Q(KN).G$) such a construction is not always available. For example, when N is abelian (that is, when G is metabelian) it is not difficult to show that any element of $Q(KN).G$ centralizing $Q(KN)$ must be a unit in $Q(KN).G$. Of course, if S has zero socle and has countable dimension over its centre then $Q(S)$ is simple and directly infinite no matter what the embeddings $S_n \rightarrow S_{n+1}$ look like (by Theorem 1.8).

CHAPTER 5

RESIDUALLY FINITE GROUP ALGEBRAS

SUMMARY

In this chapter we study the ideals of the MRQ ring of a prime nonsingular group algebra KG , and apply our results to the case where G is a residually finite group.

We consider, in §1, an arbitrary regular right self-injective ring R . With a slight change in the definition of Goodearl's sets $H(\aleph)$, we prove (Theorem 4) that these sets are in fact characteristic ideals of R .

In §2 we turn to the MRQ ring R of a nonsingular group algebra KG , and we use Theorem 4 to study the interaction between the ideals $H(\aleph)$ of R and the normal subgroups of G . We prove, for instance:

Theorem 6. Suppose KG is nonsingular and N is a normal subgroup of G . If \aleph is an infinite cardinal number then $H_{Q(KG)}(\aleph) \neq 0$ implies that $H_{Q(KN)}(\aleph) \neq 0$.

The results we obtain in §2 enable us to imitate the methods of Chapter 3, §4 for residually finite factor groups. We thus prove (in §3) the main result of this chapter:

Corollary 12. Suppose G is a residually finite group such that KG is prime nonsingular. Then $Q(KG)$ is a simple ring.

1. IDEALS IN REGULAR RIGHT SELF-INJECTIVE RINGS

Suppose KG is a prime, nonsingular group algebra with MRQ ring Q . We would like to know how the ideals of Q , the $H(\aleph)$ defined by Goodearl (see Theorem 1.1), affect the normal subgroups of G and the corresponding MRQ rings. We hope for results similar to those in Chapter 3 for $\text{soc } Q$ and for this we would like to be able to talk of $H(\aleph)$ in the MRQ rings of the subgroup-algebras. However, as a normal subgroup of G need not be prime, we must first consider whether it makes sense to talk of "the ideals $H(\aleph)$ " in regular, right self-injective rings which are not prime. We tackle this problem here.

For our $H(\aleph)$ to be ideals in arbitrary, regular, right self-injective rings we must make a small adjustment to the definition used by Goodearl in the prime case (see Theorem 1.1):

Definition. If R is a regular, right self-injective ring and \aleph is an infinite cardinal number, we write $H(\aleph) = \{r \in R : rR \text{ contains no nonzero submodules } C \cong E(\aleph(C))\}$. If there is any doubt about which ring R is intended (and there will be) we write $H_R(\aleph)$ instead.

We note that when R is prime this definition is in fact equivalent to Goodearl's in Theorem 1.1 because of Lemma 10 of Goodearl [15]. Furthermore if R is prime and $\text{soc } R \neq 0$ (so that R is a full linear ring) we have $H(\aleph_0) = \text{soc } R$.

In the general case, however, it is not immediately clear that $H(\aleph)$ need even be an ideal of R . Certainly closure under multiplication gives no trouble (compare with Corollary 3.7 of Goodearl and Boyle [17]) but we must do a little spade work for the closure under addition.

Remark. Goodearl and Boyle have, in effect, already shown that $H(\aleph_0)$ is an ideal of R : it is easy to see that $H(\aleph_0) = \{r \in R : rR \text{ is directly finite}\}$ (see, for example, the proof of Lemma 11 of Goodearl [15]) and this latter subset of R is a characteristic ideal of R by Corollary 3.7 of Goodearl and Boyle [17]. Thus in the following lemmas we restrict our attention to the case $\aleph > \aleph_0$. Even in this case, we could probably deduce these lemmas from the results in chapters XII and XIII of Goodearl and Boyle [17]. However, to do so would introduce too much machinery and obscure what is essentially a straightforward idea.

In the next three lemmas, R is a regular, right self-injective ring, A and B are nonsingular injective right R -modules, and \aleph is an infinite cardinal number such that $\aleph > \aleph_0$.

Lemma 1. There is a decomposition $A = A' \oplus A''$ where $A' \cong 2A'$ and A'' is directly finite.

Proof. Simply let A' be the injective envelope in A of the sum of a maximal family of independent submodules X of A such that each $X \cong 2X$, and let A'' be a complement to A' in A .

□

Lemma 2. Suppose $A \cong 2A$ and $B \cong 2B$. If neither A nor B contains a nonzero submodule $C \cong E(N(C))$ then neither does $A \oplus B$.

Proof. Suppose C is a submodule of $A \oplus B$ such that $C \cong E(N(C))$. By Lemma 3.4 of Goodearl and Boyle [17], we can write $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ where $C \cong A_1 \oplus B_1$. By Theorem 3.3 of [17], there is a central idempotent $u \in R$ such that $A_1 u \lesssim B_1 u$ and $B_1(1-u) \lesssim A_1(1-u)$. Now $Cu \cong E(N(Cu))$ and $Cu \cong A_1 u \oplus B_1 u \lesssim B_1 u \oplus B_1 u \lesssim Bu \oplus Bu \cong Bu$ so that, by our hypothesis about B , we have $Cu = 0$. Similarly $C(1-u) = 0$ and so $C = 0$, as required. \square

Lemma 3. Suppose A is directly finite and $B \cong 2B$. If B contains no nonzero submodules $C \cong E(N(C))$ then neither does $A \oplus B$.

Proof. Let C be a submodule of $A \oplus B$ such that $C \cong E(N(C))$ and write, as in Lemma 2, $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ where $C \cong A_1 \oplus B_1$. Since $C \cong E(N(C))$ we certainly have $C \cong 2C$ and so $A_1 \oplus B_1 \oplus A_1 \oplus B_1 \cong A_1 \oplus B_1$. As A_1 is directly finite we have (by Theorem 3.8 of Goodearl and Boyle [17]) $B_1 \oplus A_1 \oplus B_1 \cong B_1$. Hence we have both $A_1 \lesssim B_1$ and $2B_1 \lesssim B_1$. Thus $C \cong A_1 \oplus B_1 \lesssim B_1 \oplus B_1 \lesssim B_1$ and so $C = 0$, as required. \square

We are now ready for the main result of this section.

Theorem 4. Suppose R is a regular, right self-injective ring and \aleph is any infinite cardinal number.

If A and B are nonsingular, injective, right R -modules containing no nonzero submodules $C \cong E(\aleph(C))$ then $A \oplus B$ contains no such submodules.

In particular, $H(\aleph)$ is a characteristic ideal of R .

Proof. (As we noted earlier the case $\aleph = \aleph_0$ is dealt with in Theorem 3.6 and Corollary 3.7 of Goodearl and Boyle [17].) We use Lemma 1 to write $A = A' \oplus A''$ and $B = B' \oplus B''$ where $A' \cong 2A'$, $B' \cong 2B'$ and A'' and B'' are directly finite (of course, if $\aleph = \aleph_0$ we have $A' = B' = 0$). By Theorem 3.6 of [17], $A'' \oplus B''$ is directly finite. By Lemma 2 we see that $A' \oplus B'$ contains no nonzero submodules $C \cong E(\aleph(C))$. Hence, by Lemma 3, neither does $(A'' \oplus B'') \oplus (A' \oplus B') = A \oplus B$, as required. The rest of the proof follows as in Corollary 3.7 of [17]. □

Remark 1. In fact it follows easily from Theorem 4 that, for any nonsingular injective right R -module A , the set $\{a \in A : aR \text{ contains no nonzero submodules } C \cong E(\aleph(C))\}$ is a fully invariant submodule of A and coincides with $A \cdot H(\aleph)$.

Remark 2. It is not true in general (as it is in the prime case) that every ideal of R in Theorem 4 is of the form $H(\aleph)$. Indeed it is easy to find examples where R has characteristic ideals not of the form $H(\aleph)$. For instance, if K_1, K_2, \dots are fields and $R = \prod_i K_i$ then $\text{soc } R$ is a nonzero proper characteristic ideal of R but the smallest $H(\aleph)$ is $H(\aleph_0) = R$ since R is commutative.

2. IDEALS IN $Q(KG)$ AND NORMAL SUBGROUPS OF G .

Our main aim in this chapter is to show that if KG is prime nonsingular and G is residually finite then $Q(KG)$ is a simple ring. We shall prove this using basically the same methods as we used to show that $Q(KG)$ cannot be an infinite dimensional full linear ring when G is residually finite (see Corollary 3.11). However instead of working (as we did in Chapter 3) with $\text{soc } Q(KG)$, we employ the ideals $H(N)$ found in §1. In this section, therefore, we examine the interaction between these ideals $H(N)$ and the normal subgroups of G .

Proposition 5. Suppose KG is nonsingular and N is a normal subgroup of G with $[G:N] < \infty$. For any infinite cardinal number we have:

$$H_{Q(KN)}(N) \subseteq H_{Q(KG)}(N).$$

Proof. Let $Q(KG) = Q$ and $Q(KN) = R$, both of which are regular right self-injective rings since KG is nonsingular. Let T be a transversal for N in G so that, by Lemmas 3.2(b) and 3.7(a), T is a finite normalizing basis for Q over R (since $[G:N] < \infty$). Thus if $a \in H_R(N)$ we have

$$aQ = \sum_{t \in T} atR \cong \bigoplus_{t \in T} (t^{-1}at)R$$

(as R -modules). As $H_R(N)$ is a characteristic ideal of R (by Theorem 4), each $t^{-1}at \in H_R(N)$ (since the function $r \rightarrow t^{-1}rt$ is an automorphism of R by Lemma 3.2(a)). Hence, by Theorem 4, aQ has no nonzero R -submodule $C \cong E(N(C))$. Thus aQ contains no nonzero Q -submodules $C \cong N(C)$, and so $a \in H_Q(N)$ as required. □

We would like to be able to prove that (under the hypotheses of Proposition 5)

$$(H_{Q(KG)}(\aleph) \cap Q(KN)) \cdot G = H_{Q(KG)}(\aleph)$$

since we could then imitate the proof of Proposition 3.9 with $H(\aleph)$ replacing $\text{soc } Q(KG)$. Proposition 5 gives a step towards this equality (compare it with Lemma 3.8) but the trick in Lemma 3.8, using large right ideals to give the rest of the equality, is clearly not available once we allow the possibility of zero socle. Instead we adopt the following approach (inspired by Proposition 3.3).

Theorem 6. Suppose KG is nonsingular and N is a normal subgroup of G . If \aleph is an infinite cardinal number then $H_{Q(KG)}(\aleph) \neq 0$ implies that $H_{Q(KN)}(\aleph) \neq 0$.

Proof. For simplicity's sake we write $Q(KG) = Q$, $Q(KN) = R$ and $Q(KN) \cdot G = S$. As Q is a right quotient ring of S we have $S \cap H_Q(\aleph) \neq 0$. Choose some nonzero $a \in S \cap H_Q(\aleph)$ such that

$$a = a_1 x_1 + \dots + a_n x_n \quad \dots (*)$$

where, for each i ($1 \leq i \leq n$), $0 \neq a_i \in R$ and $x_i \in G$. We may suppose that $x_1 = 1$ and n is minimal for all these properties. We show that $a_1 \in H_R(\aleph)$ and that will complete the proof.

Suppose, on the contrary, that $a_1 R$ contains a nonzero right ideal $A \cong E(\aleph(A))$. Being injective, A must be a principal right ideal, say $A = a_1 r R$ where $r \in R$. By multiplying (*) on the right by r , we may assume that

$A = a_1 R$ without altering our other hypotheses (since $a_1 r + (a_2 x_2 r x_2^{-1}) x_2 + \dots + (a_n x_n r x_n^{-1}) x_n$ is still of the same form as a).

Hence there is a family B of elements of $a_1 R$ such that $|B| = \aleph$, the right ideals bR ($b \in B$) are independent, and for each $b \in B$ we have $a_1 R \lesssim bR$ (all because $a_1 R \cong E(\aleph(a_1 R))$). We shall construct a similar family of elements of aQ and thus derive the required contradiction.

As R is regular and, for each $b \in R$, $a_1 R \lesssim bR$ there are, for each such b , elements $s_b, t_b \in R$ such that $a_1 = s_b b t_b$ (if $f: a_1 R \rightarrow bR$ is the R -monomorphism then $f(a_1 R)$ is a direct summand of R and so there is a left multiplication $s_b: f(a_1 R) \rightarrow a_1 R$ such that $s_b(f(a_1)) = a_1$). As each $b \in a_1 R$ there are elements $u_b \in R$ (for each $b \in B$) such that $b = a_1 u_b$. Consider the family $\{a u_b: b \in B\}$ of elements of aQ .

For each $b \in B$ we have

$$s_b(a u_b) t_b = [s_b(a_1 u_b) t_b] + [s_b a_2 (x_2 u_b t_b x_2^{-1})] x_2 + \dots \\ \dots + [s_b a_n (x_n u_b t_b x_n^{-1})] x_n$$

where each expression in square brackets is in R . Since $s_b(a_1 u_b) t_b = s_b b t_b = a_1$ and since $(s_b(a u_b) t_b - a)$ still lies in $S \cap H_Q(\aleph)$, the minimality of n shows that $a = s_b(a u_b) t_b$. As Q is regular we thus have $aQ \lesssim a u_b Q$ for each $b \in B$.

Furthermore, for each $b \in B$,

$$a u_b = a_1 u_b + a_2 (x_2 u_b x_2^{-1}) x_2 + \dots + a_n (x_n u_b x_n^{-1}) x_n$$

so that, by the minimality of n , we have

$$\ell_R(a_1 u_b) = \ell_R(a_2 x_2 u_b x_2^{-1}) = \dots = \ell_R(a_n x_n u_b x_n^{-1}).$$

As R is regular it follows that

$$a_1 u_b R = a_2 x_2 u_b x_2^{-1} R = \dots = a_n x_n u_b x_n^{-1} R$$

and, in particular, for each $b \in B$, we get $au_b \in a_1u_bQ = bQ$. But the right ideals bR ($b \in B$) are independent. As S is free over R , and as Q is a right quotient ring of S , it follows that the right ideals bQ ($b \in B$) are also independent. Hence so are the right ideals au_bQ ($b \in B$).

Thus we have proved that

$$\aleph(aQ) \lesssim \bigoplus_{b \in B} au_bQ \subseteq aQ$$

and so $aQ \cong E(\aleph(aQ))$ which contradicts the fact that $a \in H_Q(\aleph)$. Hence $a_1 \in H_R(\aleph)$ as required. \square

We can now prove the analogue of Lemma 3.8 which we have been seeking.

Corollary 7. Suppose KG is prime nonsingular and N is a normal subgroup of G such that $[G:N] < \infty$. If \aleph is the first infinite cardinal number for which $H_{Q(KG)}(\aleph) \neq 0$ then

$$(H_{Q(KG)}(\aleph) \cap Q(KN)) \cdot G = H_{Q(KG)}(\aleph).$$

Proof. As usual write $Q(KG) = Q$ and $Q(KN) = R$. By our hypothesis Q is prime, regular and right self-injective. Hence (by Theorem 1.1) the ideals of Q are well-ordered and of the form $H_Q(\aleph)$. Thus if \aleph is the first infinite cardinal number for which $H_Q(\aleph) \neq 0$ then $H_Q(\aleph)$ is the unique minimal (nonzero) ideal of Q . By Theorem 6 we have $H_R(\aleph) \neq 0$ and so, by Proposition 5, it follows that $H_Q(\aleph) \cap R \neq 0$. As $[G:N] < \infty$ we have $Q(KG) = Q(KN) \cdot G$ (by Lemma 3.7(a)) and so $(H_Q(\aleph) \cap R) \cdot G$ is a nonzero ideal of Q . By the minimality of $H_Q(\aleph)$ we get $H_Q(\aleph) = (H_Q(\aleph) \cap R) \cdot G$ as required. \square

One consequence of Corollary 7 is:

Proposition 8. Suppose KG is prime nonsingular and $N \trianglelefteq G$ such that $[G : N] < \infty$. Then $Q(KG)$ is

- (i) simple Artinian, or
 - (ii) simple and directly infinite, or
 - (iii) simple, directly finite but not Artinian,
- if and only if the same statement is true of $Q(KN)$.

Proof. Since $\Delta^+(G) = 1$ and $[G : N] < \infty$ we see that $\Delta^+(N) = 1$ and so KN is prime too. If $Q(KN) = R$ is simple then Corollary 7 ensures that $Q(KG) = Q$ is simple as well. Suppose, conversely, that Q is simple. If $H_R(N)$ is the minimal (nonzero) ideal of R then, as $[G : N] < \infty$, $H_R(N) \cdot G$ is a nonzero ideal of Q and so must equal Q . Hence $H_R(N) = R$ and R is simple too. Thus Q is simple if and only if R is simple.

Case (i) now follows from Lemma 3.8. On the other hand if R is directly infinite so is Q (by Lemma 3.1), while if R is directly finite (that is, if $R = H_R(N_0)$) then so is Q (since $H_Q(N_0)$ contains $H_R(N_0)$ by Proposition 5). Cases (ii) and (iii) now follow easily. □

Thus, for instance, if N is the alternating subgroup of the restricted symmetric group G on any infinite set X then $Q(KN)$ is simple and directly infinite (by Proposition 8 and by Theorem 2.12).

Before we apply Corollary 7 to residually finite factor groups, we note another consequence of Theorem 6, this time

for MRQ rings of locally finite group algebras. This result improves Theorem 2.9.

Corollary 9. Suppose G is a locally finite group for which KG is prime nonsingular. If G contains a nontrivial, prime, subnormal CC-group N then $Q(KG)$ is simple and directly infinite.

Proof. By Theorem 2.9, we know that $Q(KN) = R$ is simple and directly infinite. As N is subnormal in G , Lemma 3.1 tells us that $Q(KG) = Q$ is directly infinite too. If Q is not simple then, by Proposition 2.13, we have $H_Q(N_0) \neq 0$. By Theorem 6 and induction on the length of a finite series between N and G , we have $H_R(N_0) \neq 0$ which is impossible since R is simple and directly infinite. Hence Q is simple. \square

Even when N is not prime in Corollary 9, we can still say a little about $Q(KG)$.

Proposition 10. Suppose KG is prime nonsingular. If N is a nontrivial, locally finite CC-group subnormal in G then $Q(KG)$ is directly infinite.

Proof. If $Q(KG)$ is directly finite then KN is prime (by Proposition 4.12) and so $Q(KN)$ is (simple and) directly infinite (by Theorem 2.9). But, by Lemma 3.1, $Q(KN)$ is directly finite (since $Q(KG)$ is, and since N is subnormal in G). This contradiction shows that $Q(KG)$ is directly infinite. \square

3. RESIDUALLY FINITE FACTOR GROUPS.

In view of Corollary 7 above, we are now ready to prove the following analogue of Proposition 3.9.

Theorem 11. Suppose KG is prime nonsingular and N is a normal subgroup of G for which G/N is residually finite. For any nonzero ideal I of $Q(KG)$ we have $I \cap Q(KN) \neq 0$.

Proof. As the ideals of $Q(KG)$ are well-ordered by inclusion (by Theorem 1.1), we need only consider the minimal nonzero ideal, $H(N)$ say, of $Q(KG)$. As $Q(KG)$ is a right quotient ring of $Q(KN) \cdot G$ we can choose some nonzero $a \in H(N) \cap Q(KN) \cdot G$. Write $a = a_1 g_1 + \dots + a_n g_n$ where each $a_i \in Q(KN)$ and the cosets Ng_1, \dots, Ng_n of N are distinct. Since G/N is residually finite there is a normal subgroup M of G containing N such that $[G:M] < \infty$ and the cosets Mg_1, \dots, Mg_n are distinct. By Corollary 7 we have

$$a \in H(N) = (H(N) \cap Q(KM)) \cdot G.$$

If we equate the components for a given by this with the components of the expression $a_1 g_1 + \dots + a_n g_n$ (we can do this because the cosets Mg_1, \dots, Mg_n are distinct; see Lemma 3.2(b)) we find that each $a_i \in H(N)$. Hence, as each $a_i \in Q(KN)$ and as $a \neq 0$, we have $H(N) \cap Q(KN) \neq 0$ and the result follows. □

Remark. Unlike Proposition 3.9, Theorem 11 is not true if we drop the primeness of KG (when $N = 1$, requiring that $I \cap Q(KN) \neq 0$ for each nonzero ideal I of $Q(KG)$ is the same

as requiring that $Q(KG)$ be simple and so in this case KG must be prime). Nor is Theorem 11 extendable to an intersection theorem for all nonzero ideals of the group algebra KG (such as the intersection theorems we met in Chapter 4 were): putting $N = 1$ and considering any nontrivial (residually finite) group G we see that ωG is a nonzero ideal of KG for which $\omega G \cap KN = 0$. However, as Proposition 3.9 shows, we can drop the primeness in Theorem 11 if we consider only the ideal $I = \text{soc } Q(KG)$.

An immediate consequence of Theorem 11 is:

Corollary 12. Suppose G is a residually finite group such that KG is prime nonsingular. Then $Q(KG)$ is a simple ring. □

Remark 1. As in the case where G is soluble it is easy to give examples of residually finite groups G for which $Q(KG)$ is simple Artinian (for instance, the infinite cyclic group is residually finite) or simple and directly infinite (if G is any noncyclic free group then KG is a non-Ore domain and so $Q(KG)$ is simple and directly infinite). I do not know if it is possible for $Q(KG)$ to be an $SP(\infty)$ ring.

Remark 2. If G is a noncyclic free group then the domain KG contains a family of $|G|$ independent nonzero right ideals (consider, for instance, the right ideals $\omega(< x >)$ as x runs through a basis for G - see Formanek's example in [13]). Hence any cardinal number \aleph_α , where α is a non-limit ordinal,

can be the first N such that $H(N) \neq 0$ in $Q(KG)$: simply let $|G| = N_{\alpha-1}$. Choosing $G = \mathbb{Z}$ shows that N_0 can also be the first such N .

Remark 3. Again KG need not be strongly prime in Corollary 12 even though $Q(KG)$ is a simple ring. In fact if A and B are residually finite, torsion, abelian groups such that A is nontrivial and has no elements of order $\text{char } K$, and B is infinite, then $G = A \setminus B$ is a residually finite group (by Theorem 3.2 of Gruenberg [21]) for which KG is prime, nonsingular but not strongly prime (since G is itself locally finite).

A more general, but still immediate, consequence of Theorem 11 is:

Corollary 13. Suppose KG is prime nonsingular and N is a normal subgroup of G such that G/N is residually finite and $Q(KN)$ is simple. Then $Q(KG)$ is simple too. \square

This could suggest an inductive approach similar to that used in Chapter 3 for the case where KG had, in addition, uniform right ideals. Along the same lines we have the following fairly straightforward result which was pointed out to me by K.A. Brown.

Proposition 14. Suppose KG is prime nonsingular and N is a normal subgroup of G such that $Q(KN)$ is simple and, in the notation of Chapter 3, §4, $G/N \in \mathcal{V}$. Then $Q(KG)$ is simple too. \square

Thus, for instance, G/N could be any torsion-free locally nilpotent group.

However, in each case, the assumption that $Q(KN)$ be simple is not one that follows naturally from our hypotheses (as it did in Chapter 3, where all subnormal subgroups of G were at least prime), and we cannot expect this approach to be as fruitful as it was in Chapter 3.

Finally, as a sort of omnibus edition of the results of the last two chapters, we have:

Proposition 15. Suppose KG is prime nonsingular and $N \trianglelefteq G$ such that N is soluble and G/N is residually finite. Then $Q(KG)$ is a simple ring.

Proof. Let I be a nonzero ideal of $Q(KG)$. By Theorem 11 we have $I \cap Q(KN) \neq 0$ (since G/N is residually finite) and hence $I \cap KN \neq 0$. Let H be the Zaleskiĭ subgroup of the soluble group N . By the intersection theorem of Zaleskiĭ [54] we thus have $I \cap KH \neq 0$. Now H is an FC-group and, being characteristic in N , is normal in G . Hence if KH has no uniform right ideals, Theorem 4.3 implies that $I = Q(KG)$. On the other hand, if KH does have a uniform right ideal then Lemma 4.9 implies that H has no nontrivial, locally finite, normal subgroups. Being an FC-group, H would then be torsion-free abelian and so $I \cap KH \neq 0$ implies that I contains a unit of $Q(KG)$. Thus in either case $I = Q(KG)$ and so $Q(KG)$ is simple. \square

OPEN QUESTIONS

We mention here some of the unsolved problems related to the work in this thesis.

We have already discussed, in Chapter 3, our first query:

Question 1. Is $Q(KG)$ ever an infinite dimensional full linear ring?

In their paper [23], Handelman and Lawrence suggest:

Question 2. If G has no nontrivial locally finite normal subgroups, is KG a strongly prime ring?

We showed in Theorem 3.5 that an affirmative answer to Question 2 implies a negative answer to Question 1. Also related to Question 1 is the following special case of a question asked by Richardson in [48]:

Question 3. Is there a prime group G which is not locally finite and a field K such that $\text{soc } KG \neq 0$?

Richardson conjectures that the answer to this is 'no' but if there is such a group G and such a field K then $Q(KG)$ gives an affirmative answer to Question 1 (see Chapter 3, §4).

A more general form of Question 1 is suggested by the results of Chapter 4 and 5:

Question 4. Suppose KG is prime nonsingular. Is $Q(KG)$ ever non-simple?

When G is soluble or residually finite the answer is 'no' (by Corollaries 4.11 and 5.12). Furthermore if G is locally finite then $Q(KG)$ has at most one nonzero proper ideal (by Proposition 2.13).

Another problem raised by the results of Chapters 4 and 5 is:

Question 5. Suppose KG is prime nonsingular. Is $Q(KG)$ ever an $SP(\infty)$ ring?

In particular we wonder if $Q(KG)$ can be $SP(\infty)$ when G is locally finite (see Proposition 2.13).

We discussed Question 5 in the case where G is a soluble group in Chapter 4, §2, and we showed there that a related question is:

Question 6. Let S be a ring (with identity) containing simple Artinian subrings $S_1 \subseteq S_2 \subseteq \dots$ where $S = \bigcup_n S_n$. Is $Q(S)$ ever an $SP(\infty)$ ring?

TABLES

In the following tables we summarize what is currently known about maximal quotient rings of prime nonsingular group algebras. We display both the progress made in this thesis and the progress still to be made.

How to read the tables

We read the situation illustrated at right as follows: if G has property (1) then Q has property (2) if and only if the additional conditions (3) are also satisfied.

$G \downarrow \begin{array}{c} Q \\ \rightarrow \end{array}$		(2)
(1)		(3)

If no precise condition (3) is known then examples where G satisfies (1) and Q satisfies (2) are given in the square (3). If no such example is known the phrase 'not known' is put in the square instead.

Further elaboration (such as justification, sources for the stated result, or comments) is provided in the notes after the tables. The labels of these notes may be found at the appropriate place in the tables.

TABLE 1

THE GENERAL CASE: KG is prime nonsingular, $Q = Q_{\max}(KG)$

$\begin{array}{c} Q \longrightarrow \\ G \downarrow \end{array}$	Q is a simple ring			Q is not simple	
	Simple Artinian. $SP(n)$ for $n > 1$ or a division ring	Simple, directly finite but not Artinian. $SP(\infty)$	Simple directly infinite. $SP(1)$, not a division ring.	Infinite dimensional full linear ring	Non-simple with zero socle.
All conjugacy classes countable ¹ (see explanatory notes)	eg: $G = Z\{Z_p$	Never	When KG has no uniform right ideals	Not known	Never
Locally finite ²	Never	Not known	eg: G is the symmetric group on an infinite set	Never	Not known
Soluble ³	eg: G is torsion- free abelian	Not known	eg: G is locally finite	Never	Never
Residually finite ⁴	eg: G is poly- cyclic-by-finite	Not known	eg: $\text{char } K \neq p$ and $G = Z_p\{Z$.	Never	Never
Linear	eg: G is abelian -by-finite	Not known	eg: G is the free group on two generators	Not if $\text{char } K = 0$. ⁵ Otherwise unknown	Not known
Free products ⁶	When $G = Z_2 * Z_2$	Never	Except when $G = Z_2 * Z_2$	Never	Never
Locally nilpotent ⁷	When G is torsion-free	Not if G satisfies the normalizer condition. Other- wise unknown.	eg: $G = Z_p\{Z_{p^\infty}$ and $\text{char } K \neq p$.	Never	Not if G satisfies the normalizer condition. Other- wise unknown.
Nontrivial normal CC-subgroup N	eg: see first row	Not if KN has no uniform right ideals. Other- wise unknown. ⁸	eg: see first row	Not known	Not known
	Q is directly finite		Q is directly infinite		

TABLE 2

THE LOCALLY FINITE CASE: KG is prime nonsingular, $Q = Q_{\max}(KG)$, G is nontrivial and locally finite, $H(\mathcal{N})$ the ideals of Q defined by Goodearl.

Properties of Q Properties of G	Nonzero socle (includes simple Artinian, and infinite dimensional, full linear rings).	Simple, directly finite, $SP(\infty)$ $H(\mathcal{N}_0) = Q$.	Not simple, directly infinite, zero socle. $0 \subset H(\mathcal{N}_0) \subset Q$	Simple, directly infinite, $SP(1)$, not a division ring. $H(\mathcal{N}_0) = 0$ and $H(\mathcal{N}_1) = Q$	$H(\mathcal{N}_1) \subset Q$ and Q has zero socle. ¹
All conjugacy classes countable ⁴	Never ² (since KG contains no uniform right ideals)	Never	Never	Always	Never ³ (since every family of independent non- zero right ideals of KG is countable)
Soluble ⁵		Never	Never	Always	
Restricted symmetric group on an infinite set, or its alternating subgroup ⁶		Never	Never	Always	
Residually finite ⁷		Not known	Never	eg: $G = \bigcup_p \bigcup_q \mathbb{Z}_p \mathbb{Z}_q$ and $\text{char } K \neq p$, (I an infinite set)	
Linear ⁸		Never	Never	eg: $G = \text{PSL}(n, F)$ where $n \geq 2$, F an infinite locally finite field.	
Nontrivial normal CC-subgroup ⁹		Never	Not known	eg: see first row	
General case: (including simple, locally soluble, or even locally nil- potent groups)		Not known	Not known	eg: see above	

TABLE 3

THE SOLUBLE CASE: G is soluble, KG is prime nonsingular, $Q = Q_{\max}(KG)$

Properties of Q Properties of G	Simple Artinian, $SP(n)$ where $n > 1$ or a division ring	Simple, directly finite, but not Artinian $SP(\infty)$	Simple, directly infinite. $SP(1)$ but not a division ring	Infinite dimensional full linear ring	Not simple and zero socle.
Nontrivial locally finite group ¹	Never	Never	Always (eg: $\text{char } K \neq p$ and $G = \mathbb{Z}_p \wr \mathbb{Z}_{p^\infty}$)	Never	Never
Nontrivial locally finite normal subgroup	Never	Never	Always ²	Never	Never
No nontrivial, locally finite, normal subgroups. (i.e. The Zalesskii subgroup of G is torsion-free abelian) ³	eg: G is polycyclic- by-finite, or abelian-by-finite; or KG is a domain. (See [3] for the most general known result.)	Not known (guess: never) ⁴	eg: G has only countable conjugacy classes and an infinite locally finite subgroup (say: $G = \mathbb{Z} \wr \mathbb{Z}_{p^\infty}$) or, more generally, G has only countable conjugacy classes and KG has no uniform right ideals.	Never	Never

EXPLANATORY NOTES FOR THE TABLES

Notes for Table 1.

1. The statements in this row are just Theorem 1.13.
2. See Table 2 for a more detailed coverage of the locally finite case.
3. See Table 3 for more details of the soluble case.
4. Q must be simple by Corollary 5.12.
5. See Proposition 3.21.
6. If G is a nontrivial free product then KG is strongly prime (by Proposition III.3 of Handelman and Lawrence [23]) and so Q is simple. Furthermore, unless $G = Z_2 * Z_2$ (the infinite dihedral group), G contains a noncyclic free subgroup and Q is directly infinite (Lemma 3.20).
7. See Proposition 4.4 for groups satisfying the normalizer condition. The proof of Proposition 4.4 also shows that, when G is locally nilpotent, KG is prime non-singular with uniform right ideals if and only if G is torsion-free if and only if KG is an Ore domain.
8. See Theorem 4.3.

Notes for Table 2.

1. This case includes those simple, directly infinite Q for which the first \aleph such that $H(\aleph) = Q$ satisfies $\aleph > \aleph_1$ as well as those non-simple Q with zero socle which have more than one nonzero proper ideal.

2. See Proposition 2.8.
3. See Proposition 2.13.
4. See Theorem 2.9.
5. See Table 3 for the soluble case.
6. See Theorem 2.12.
7. Q is simple by Corollary 5.12. In the example where Q is simple and directly infinite, p and q are any prime numbers and G is residually finite by Theorem 3.2 of Gruenberg [21] (that Q is in fact simple and directly infinite follows because G is soluble and locally finite - see Table 3).
8. This case is not proved in the main body of the thesis so we sketch a proof here.

For any group G let $\Gamma(G) = \{g \in G : [G : C_G(g)] \leq \aleph_0\}$. We call $\Gamma(G)$ the CC-centre of G (this is analogous to the FC-centre of a group - see p.121 of Robinson [49]). Then $\Gamma(G)$ is a characteristic subgroup of G . We say that G is CC-hypercentral if G has an ascending series of normal subgroups $1 = G_0 \subseteq G_1 \subseteq \dots \subseteq G_\alpha \subseteq \dots \subseteq G$ such that, for each ordinal α , we have $G_{\alpha+1}/G_\alpha \subseteq \Gamma(G/G_\alpha)$. For example, any periodic linear group is CC-hypercentral since, by (9.5) of Wehrfritz [53], such a group is nilpotent-by-countable.

Now if G is any CC-hypercentral group and if $N = \Gamma(G)$ then we can imitate the proof of Lemma 8 of Zalesskii [54] to show that, for any nonzero ideal I of KG , we have $I \cap KN \neq 0$ (see Chapter 4, §1, for the

sort of imitation required). Since N is a CC-group we can use Theorem 4.3 and Lemma 4.9 to deduce the following result:

Theorem. Suppose G is a CC-hypercentral group with CC-centre N . If KG is prime nonsingular and N is locally finite then $Q(KG)$ is simple and directly infinite. □

Finally, the group $G = \text{PSL}(n, F)$, where $n \geq 2$ and F is an infinite locally finite field, is a simple, periodic, linear group (simple because $\text{PSL}(n, F')$ is a simple group for any $n \geq 2$ and any finite field F' with $|F'| \geq 4$, periodic since F is locally finite, linear because of Theorem 6.2 of Wehrfritz [53]). Thus if K is a field of zero characteristic, KG is prime and nonsingular and so Q is simple and directly infinite.

9. See Theorem 4.3 and Lemma 4.9.

Notes for Table 3.

1. This is a special case of the second row.
2. See Proposition 4.8 and Theorem 4.10.
3. See Proposition 4.8 and Corollary 4.11.
4. See Chapter 4, §2, for a discussion of this possibility.

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